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DECOMPOSITION TECHNIQUES FOR FINITE SEMIGROUPS, USING CATEGORIES II

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Dans cet article, nous étudions de manière systématique la décomposition de morphismes de semigroupes finis $\theta : S \to T$ de la forme $\theta = \varphi\pi$, où $\varphi : S < V_n \wr (V_{n-1} \wr \cdots (V_1 \wr T) \cdots)$. Ici, les V_i sont des monoïdes, $\wr$ représente soit le produit semidirect, soit le product semidirect bilatèral et π est la projection canonique de $V_n \wr (V_{n-1} \wr \cdots (V_1 \wr T) \cdots)$ sur T. Ces résultats sont ensuite raffinés pour des classes particulières de morphismes, et en particulier pour les morphismes apériodiques et les **LG**-morphismes.

In this paper we study in a systematic fashion the decomposition of semigroup morphisms $\theta : S \to T$ with S and T finite, in the form $\theta = \varphi\pi$, where $\varphi : S < V_n \wr (V_{n-1} \wr \cdots (V_1 \wr T) \cdots)$. Here, the V_i's are monoids, $\wr$ denotes either the semidirect or the 2-sided semidirect product, and π denotes the canonical projection of $V_n \wr (V_{n-1} \wr \cdots (V_1 \wr T) \cdots)$ onto T. These results are then refined for special classes of morphisms, and in particular for aperiodic morphisms and **LG**-morphisms.

Introduction

All the semigroups considered here are finite. In [5] the first author introduced *maximal proper surmorphisms*, or m.p.s.'s, under a slightly different name, meaning non-factorizable semigroup surmorphisms, and first proved the basic properties including a classification. In part I [10], we gave a more detailed and complete classification of non-factorizable semigroup morphisms $\theta : S \to T$ or m.p.s.'s. The reader is referred to [10] for the notations and results relative to this classification, as well as for the other undefined concepts.

In this paper, for each of the classes of m.p.s.'s, we shall characterize classes of monoids V such that there exists an injective relational morphism (division [3, 14, 15])

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$\varphi : S < V \wr T$ for which $\theta = \varphi\pi$. Here $\wr$ denotes either the semidirect or the 2-sided semidirect product and π denotes the canonical projection of $V \wr T$ onto T. These results are then extended to larger classes of relational morphisms θ. In particular, it is proved that aperiodic morphisms can be decomposed by a sequence of 2-sided semidirect products by semilattices (J_1), and **LG**-morphisms by a sequence of 2-sided semidirect products by groups. Recall that a surmorphism θ is regular iff $\theta(s)$ is regular iff s is regular. Also, we shall discuss the decomposition of regular vs. non-regular **LG**-morphisms: in the non-regular case it turns out that the use of 2-sided semidirect products gives rise to decomposition results that could not be obtained by the sole use of semidirect or reverse semidirect products.

These decomposition results rely heavily on the results developed by Tilson in [15] and Tilson and Rhodes in [9] relative to the derived category and the kernel of a relational morphism. Some of the applications were announced in [7]. These results and others were also in [16].

The paper is divided as follows: Section 1 is devoted to the statement of various known results that will be used in the sequel: we deal here with definitions of varieties of categories by Thérien, Tilson [12, 13, 15], semidirect product and 2-sided semidirect product of semigroups [9, 15]; the definitions and properties of the derived category by Tilson [15]; the kernel by Tilson and Rhodes in [9] of a relational morphism; and a few properties of expansions: Rhodes expansion and the related Stiffler and Karnofsky expansions. In Section 2, these results are put to work and we obtain decomposition results for each class of m.p.s.'s. Finally, Section 3 contains the applications of this study of m.p.s.'s to certain classes of relational morphisms, and in particular to aperiodic morphisms, **LG**-morphisms, regular **LG**-morphisms, and for morphisms that are injective on $\mathscr{R}$-classes.

1. Categories, products, kernels and expansions

1.1. Varieties of finite categories

The study of categories as a generalization of monoids was initiated by Tilson, Margolis and Pin in [15, 4] and extended in Rhodes–Tilson [9]. It was shown how categories help solve decomposition problems for semigroups. We shall review the basic definitions concerning them.

A *semigroupoid* C is given by a non-empty set of objects Ob_C, and, for all $c, d \in \mathrm{Ob}_C$, by sets $C(c,d)$ of arrows. Also, a binary operation is given, for each $c, d, e \in \mathrm{Ob}_C$, from $C(c,d) \times C(d,e)$ into $C(c,e)$. This operation is required to be associative, i.e., if x, y and z are arrows of C, either $(xy)z = x(yz)$ and both terms are defined, or both terms are undefined.

C is a *category* if each base semigroup $C(c)$ ($c \in C$) has a unit 1_c that is also a left unit for $C(c,c')$ and a right unit for $C(c',c)$ for all $c' \in \mathrm{Ob}_C$. It is clear that one-object semigroupoids (resp. categories) are semigroups (resp. monoids).

The notions of relational morphism and division of semigroups are extended to semigroupoids in the following way (see [15]). Let C and D be semigroupoids. A *relational morphism* $\varphi: C \to D$ consists of an object function $\varphi: \mathrm{Ob}_C \to \mathrm{Ob}_D$ and of a family of relations $\varphi: C(c,c') \to D(c\varphi, c'\varphi)$ $(c,c' \in \mathrm{Ob}_C)$ such that if x, x' are composable arrows of C, then $(x\varphi)(x'\varphi) \subseteq (xx')\varphi$. φ is said to be *injective* or a *division* (which we denote by $\varphi: C < D$) if, furthermore, for any two arrows x, y of $C(c,c')$ $(c,c' \in \mathrm{Ob}_C)$, $x\varphi \cap y\varphi \neq \emptyset$ implies $x=y$. When considering categories we require that identities relate to identities, etc., see [15].

A *C-variety* is a class of finite categories closed under division and *finite* direct product (see [15, 12, 13]). If V is a C-variety, then the subclass V_M of all one-object categories in V is an M-variety. Conversely, let W be an M-variety. Define $\mathbf{gW}$ as the class of all categories that divide a monoid in W and $\mathbf{lW}$ as the class of all categories all of whose divisors, if they are monoids, are in W, i.e., $C \in \mathbf{lW}$ iff $\forall\, v \in \mathrm{obj}(C)$, $C(v) \in W$. $\mathbf{gW}$ and $\mathbf{lW}$ are C-varieties respectively said to be *globally* and *locally induced* by W. Clearly $\mathbf{gW} \subseteq \mathbf{lW}$. We have [15, 12, 13] the very important

Proposition 1.1. *Let W be an M-variety. I denotes the trivial M-variety.*

(1) $(\mathbf{gW})_M = (\mathbf{lW})_M = W$.

(2) *If V is a C-variety and $W = V_M$, then* $\mathbf{gW} \subseteq V \subseteq \mathbf{lW}$.

(3) *If W is non-trivial, then* $\mathbf{gI} \subset \mathbf{lI} \subseteq \mathbf{gW}$.

(4) *If H is any non-trivial G-variety, then* $\mathbf{gH} = \mathbf{lH}$.

(5) $\mathbf{gJ}_1 = \mathbf{lJ}_1$ *where J_1 denotes the variety of semilattices (i.e. idempotent and commutative semigroups).* □

1.2. Wreath product and 2-sided product

Wreath products were introduced into semigroup theory via the Krohn–Rhodes theorem, see [3]. Let us recall the definitions. Let S and T be semigroups. The *wreath product* $S \circ T$ is the set $S^{T^{\bullet}} \times T$ with product given by $(f,t)(f',t') = (g,tt')$ and $g(u) = f(u)f'(ut)$ for all $u \in T^{\bullet}$. The *2-sided product* $S \circ\circ T$ is the set $S^{T^{\bullet} \times T^{\bullet}}$ with product given by $(f,t)(f',t') = (g,tt')$ and $g(u,v) = f(u,t'v)f'(ut,v)$ for all $u, v \in T^{\bullet}$. See [9].

Associated to these products, *semidirect products* are defined. For the sake of clarity following [3], when semidirect products are considered, we shall usually write the law of S additively (without assuming commutativity). If a left action of T on S is given, $S * T$ is the set $S \times T$ with product $(s,t)(s',t') = (s + (t \cdot s'), tt')$. If commuting right and left actions of T on S are given, $S ** T$ is the set $S \times T$ with product $(s,t)(s',t') = ((s \cdot t') + (t \cdot s'), tt')$. We say that a semidirect product or a 2-sided semidirect product is *unitary* if the actions of T on S satisfy

$$1_T \cdot s = s \cdot 1_T = s \quad \text{for all } s \in S, \text{ if } T \text{ is a monoid}$$

and

$$t \cdot 0_S = 0_S \cdot t = 0_S \quad \text{for all } t \in T \text{ if } S \text{ is a monoid.}$$

Also we define the *reverse products* by the following formulae:

$$T \circ_{\mathrm{r}} S = (S^{\mathrm{r}} \circ T^{\mathrm{r}})^{\mathrm{r}}, \qquad T *_{\mathrm{r}} S = (S^{\mathrm{r}} * T^{\mathrm{r}})^{\mathrm{r}}.$$

Note that $(S \circ\circ T)^{\mathrm{r}} = S^{\mathrm{r}} \circ\circ T^{\mathrm{r}}$ and $(S ** T)^{\mathrm{r}} = S^{\mathrm{r}} ** T^{\mathrm{r}}$.

Classical properties of the wreath product are listed in [3] and properties of the 2-sided products are to be found in [9]. Results about 2-sided products are also proved in [16]. In particular, let us note that $S \circ T = S^{T^I} * T$ and $S \circ\circ T = S^{T^I \times T^I}$. It is easy to check that direct and semidirect products are special cases of 2-sided semidirect products, and that $S \circ T$ and $T \circ_{\mathrm{r}} S$ are divisors of $S \circ\circ T$. Also, we have $S ** T \leq T *_{\mathrm{r}} (S * T)$, $S ** T \leq (T *_{\mathrm{r}} S) * T$, $S \circ\circ T \leq T \circ_{\mathrm{r}} (S \circ T)$ and $S \circ\circ T \leq (T \circ_{\mathrm{r}} S) \circ T$.

These products induce an operation on $\boldsymbol{S}$- and $\boldsymbol{M}$-varieties. Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be varieties. $\boldsymbol{V} * \boldsymbol{W}$ denotes the variety (it is an $\boldsymbol{S}$-variety if either $\boldsymbol{V}$ or $\boldsymbol{W}$ is one, an $\boldsymbol{M}$-variety otherwise) generated by all unitary products $V * W$ with V in $\boldsymbol{V}$ and W in $\boldsymbol{W}$. In fact, we have [3].

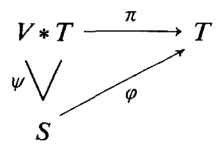
Proposition 1.2. *If $\boldsymbol{V}$ and $\boldsymbol{W}$ are $\boldsymbol{S}$-varieties, $S \in \boldsymbol{V} * \boldsymbol{W}$ iff S divides some semidirect product $V * W$ with V in $\boldsymbol{W}$ and W in $\boldsymbol{W}$. If one of $\boldsymbol{V}$ and $\boldsymbol{W}$ is an $\boldsymbol{M}$-variety, $S \in \boldsymbol{V} * \boldsymbol{W}$ iff S divides some unitary semidirect product $V * W$ with V in $\boldsymbol{V}$ and W in $\boldsymbol{W}$.* □

The definitions of the varieties $\boldsymbol{W} *_{\mathrm{r}} \boldsymbol{V}$ and $\boldsymbol{V} ** \boldsymbol{W}$ are similar and the analogue of Proposition 1.2 also holds for them [9, 16]. As a consequence of the properties of the products mentioned above, we have $\boldsymbol{W} *_{\mathrm{r}} \boldsymbol{V} = (\boldsymbol{V}^{\mathrm{r}} * \boldsymbol{W}^{\mathrm{r}})^{\mathrm{r}}$, $(\boldsymbol{V} ** \boldsymbol{W})^{\mathrm{r}} = \boldsymbol{V}^{\mathrm{r}} ** \boldsymbol{W}^{\mathrm{r}}$ and $(\boldsymbol{V} * \boldsymbol{W}) \vee (\boldsymbol{W} *_{\mathrm{r}} \boldsymbol{V}) \subseteq \boldsymbol{V} ** \boldsymbol{W} \subseteq (\boldsymbol{W} *_{\mathrm{r}} (\boldsymbol{V} * \boldsymbol{W})) \cap ((\boldsymbol{W} *_{\mathrm{r}} \boldsymbol{V}) * \boldsymbol{W})$. Note that $*$ is associative on varieties [3] and that $**$ is *not*. Neither $*$ nor $**$ is associative on semigroups.

Finally, given a product $V = S * T$ (resp. $S ** T$, $T *_{\mathrm{r}} S$, $S \circ T$, $S \circ\circ T$, $T \circ_{\mathrm{r}} S$), a canonical projection π is defined from V onto T by $(s, t)\pi = t$ (resp. $(f, t)\pi = t$). π is a morphism.

1.3. Derived category of a relational morphism

The construction described in this section is due to Tilson, in [15] and in previous earlier preprints; for an exposition, see [7]. Given a relational morphism $\varphi: S \to T$ it characterizes all semigroups V such that there exists a division $\psi: S < V * T$ for which $\psi\pi = \varphi$.

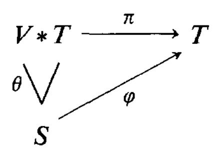


A semigroupoid is associated to the relational morphism φ as follows. We construct first a semigroupoid R_{φ} with objects T^I and arrows $R_{\varphi}(t_1, t_2) =$

$\{(s,t)\in S\times T \mid t\in s\varphi,\ t_1t=t_2\}$, $(t_1,t_2\in T^I)$. As for its product, if $(s,t)\in R_\varphi(t_1,t_2)$ and $(s',t')\in R_\varphi(t_2,t_3)$, then $(s,t)(s',t')=(ss',tt')\in R_\varphi(t_1,t_3)$. The *derived semigroupoid* of φ, D_φ, is the quotient of R_φ by the congruence Δ that preserves the objects and such that if (s,t) and (s',t') lie in some $R_\varphi(t_1,t_2)$, $(s,t)\Delta(s',t')$ iff, for all $s_1\in t_1\varphi^{-1}$, $s_1s=s_1s'$. We denote the class of $(s,t)\in R_\varphi(t_1,t_2)$ by $(t_1,[s,t])\in D_\varphi(t_1,t_2)$. Tilson in [15] proved the following important theorem which we will use numerous times.

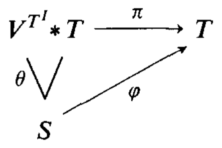
Proposition 1.3. *Let S, T, V be semigroups.*

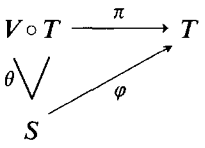
(1) *If there is a division $\theta : S < V*T$ and if $\varphi=\theta\pi$, then $D_\varphi < V$.*



(2) *If $\varphi : S\to T$ is a relational morphism and $D_\varphi < V$, then there exists a division $\theta : S < V^{T^I} * T$ with $\varphi=\theta\pi$ and $V^{T^I}*T\subseteq V\circ T^I$.*

(3) *If, further, S, T and V are monoids and $1_T\in 1_S\varphi$, then $D_\varphi < V$ implies the existence of a division $\theta : S < V\circ T$ such that $\theta\pi=\varphi$.*





Proof. We very briefly sketch a proof. For full details see [15].

(1) To construct the division, map the arrow $t_1 \xrightarrow{(s,t)} t_1t$ to t_1v, where (v,t) is an element which maps onto s.

(2) Lift $s\in S$ to $\tilde{s}\equiv(t\to i(t\xrightarrow{(s,\bar{s})} t\bar{s}),\bar{s})$ where $\bar{s}$ is chosen arbitrarily but fixed in $\varphi(s)$ and $i(\alpha)$ is an arbitrary but fixed lift to V of the arrow α of the given division. Then $\langle\tilde{s}: s\in S\rangle \twoheadrightarrow S$, given by $\tilde{s}_1\cdots\tilde{s}_n\mapsto s_1\cdots s_n$ is the required division by considering the object I. $\square$

Note the following particular case:

Proposition 1.4. *If $\varphi : G\to H$ is an onto group morphism, then D_φ divides and is divided by the group* ker(φ).

Proof. First, it is easy to check that D_φ, whose object set is H^I, divides and is divided by its subcategory determined by the objects in H. For each h_1,h_2 in H, $R_\varphi(h_1,h_2)=\{(g,h_1^{-1}h_2)\mid g\varphi=h_1^{-1}h_2\}$ is non-empty since φ is onto. In particular, R_φ is a connected groupoid (a category with inverses) where the inverse of the arrow (g,h) is (g^{-1},h^{-1}). It is well known that R_φ divides and is divided by any of its base groups. Finally, since G is a group, the congruence Δ is trivial, so that $D_\varphi=R_\varphi$. $\square$

1.4. Kernel of a relational morphism

This construction is analogous to the derived semigroupoid of a relational morphism $\varphi : S \to T$ but relative to the use of bilateral products. It was first developed by Rhodes–Tilson in [9] (see also [16]). A semigroupoid $R_{2,\varphi}$ is constructed as follows. It has object set $T^I \times T^I$ and arrow sets

$$R_{2,\varphi}((t_1,t_2),(t_1',t_2')) = \{(s,t) \in S \times T \mid t \in s\varphi,\ t_1 t = t_1',\ t_2 = t t_2'\}.$$

If $(s,t) \in R_{2,\varphi}((t_1,t_2),(t_1',t_2'))$ and $(s',t') \in R_{2,\varphi}((t_1',t_2'),(t_1'',t_2''))$, then the product $(s,t)(s',t')$ is (ss',tt') and lies in $R_{2,\varphi}((t_1,t_2),(t_1'',t_2''))$. As above, the *kernel of* φ, K_φ, is the quotient of $R_{2,\varphi}$ by a congruence Δ that preserves the objects. Two arrows (s,t) and (s',t') in $R_{2,\varphi}((t_1,t_2),(t_1',t_2'))$ are Δ-equivalent if the mappings from $t_1\varphi^{-1} \times t_2'\varphi^{-1}$ into $(t_1 t t_2')\varphi^{-1} = (t_1 t_2)\varphi^{-1} = (t_1' t_2')\varphi^{-1}$ that assign to a pair (s_1, s_2'), respectively $s_1 s s_2'$ and $s_1 s' s_2'$ are identical. We denote the class of $(s,t) \in R_{2,\varphi}((t_1,t_2), (t_1',_2))$ by $(t_1,[s,t],t_2')$.

Let us note that

Proposition 1.5. *$K_{\varphi^r} = (K_\varphi)^r$ and $K_\varphi < D_\varphi$.*

Proof. The first statement is elementary. To prove the second one, let $\mu : T^I \times T^I \to T^I$ be the projection $(t,t')\mu = t$ and, for all $d_1 = (t_1,t_1')$ and $d_2 = (t_2,t_2')$ in $T^I \times T^I$, let $\mu : K_\varphi(d_1,d_2) \to D_\varphi(t_1,t_2)$ be the relation defined by $\psi\mu = \{(t_1,[s,t]) \mid w = (t_1,[s,t],t_2')\}$. μ is multiplicative by construction. Further, μ is injective. Indeed, if $w, w' \in K_\varphi(d_1,d_2)$ and $x \in (w\mu) \cap (w'\mu)$, then there exists (s,t) and (s',t') in $S \times T$ such that $t \in s\varphi$, $t' \in s'\varphi$, $w = (t_1,[s,t],t_2')$, $w' = (t_1,[s',t'],t_2')$ and $x = (t_1,[s,t]) = (t_1,[s',t'])$. Since $(s,t)\Delta(s',t')$ in $R_\varphi(t_1,t_2)$, $s_1 s = s_1 s'$ for all s_1 in $t_1\varphi^{-1}$. So, for all s_1 in $t_1\varphi^{-1}$ and s_2' in $t_2'\varphi^{-1}$, $s_1 s s_2' = s_1 s' s_2'$. Thus $(s,t)\Delta(s',t')$ in $R_{2,\varphi}(d_1,d_2)$ and hence $w = w'$. Thus μ is a division of D_φ by K_φ. □

The analogous of Proposition 1.4 also holds.

Proposition 1.6. *If $\varphi : G \to H$ is an onto group morphism, then K_φ divides and is divided by the group* $\ker(\varphi)$.

Proof. By Propositions 1.4 and 1.5, $K_\varphi < \ker(\varphi)$. For the converse, it is enough to check that the base semigroup of K_φ at the object $(1,1)$ is $\ker(\varphi)$. □

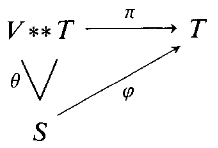
Let us finally note the following elementary lemma:

Lemma 1.7. *Let $k = (t_1,t_2)$ be an object of K_φ and $(t_1,[s,t],t_2)$ be in $K_\varphi(k)$. If for all $s_1 \in t_1\varphi^{-1}$ and $s_2 \in t_2\varphi^{-1}$, $s_1 s s_2 = s_1 s_2$, then $(t_1,[s,t],t_2)$ is the local identity of K_φ at object k.* □

Rhodes and Tilson in [9] proved the following (see also [16]).

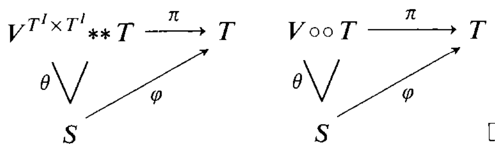
Proposition 1.8. *Let S, T and V be semigroups.*

(1) *If there exists a division $\theta : S < V ** T$ and if $\varphi = \theta\pi$, then $K_\varphi < V$.*



(2) *If $\varphi : S \to T$ is a relational morphism and $K_\varphi < V$, then there exists a division $\theta : S < V^{T^I \times T^I} ** T$ with $\varphi = \theta\pi$, and*

(3) *If, further, S, T and V are monoids and $1_T \in 1_S\varphi$, then $K_\varphi < V$ implies the existence of a division $\theta : S < V \circ\circ T$ such that $\varphi = \theta\pi$.*



□

An important first application of Proposition 1.8, that we shall use later, is the following proposition:

Proposition 1.9. *Let S be a semigroup and N be an ideal of S such that $N^2 = \{0\}$. For any non-trivial monoid M and for any large enough integer n, $S < M^n ** S/N$. If S is a monoid, n can be chosen such that $S < M^n \circ\circ S/N$.*

Proof. Let π be the canonical projection of S onto S/N. If we can prove that $K_\pi \in \mathbf{II}$, we shall be done, after Proposition 1.1(3) (applied to the M-variety generated by M), and Proposition 1.8. So we want to check that every non-empty $K_\pi(k)$ (with $k = (t_1, t_2)$ in $S/N \times S/N$) contains exactly one element, which is a local identity. Let $(t_1, [s, t], t_2) \in K_\pi(k)$: then $t_1 t = t_1$ and $t_2 = t t_2$. Let now $s_1 \in t_1\pi^{-1}$ and $s_2 \in t_2\pi^{-1}$. If $t_1 \neq 0$, then $t_1\pi^{-1} = \{s_1\}$ so that $s_1 s = s_1$, since $(s_1 s)\pi = t_1 t = t_1$. Similarly, if $t_2 \neq 0$, $s s_2 = s_2$. In both cases, $s_1 s s_2 = s_1 s_2$. If $t_1 = t_2 = 0$, then $s_1, s_2 \in N$, so that $s_1 s s_2$ and $s_1 s_2$ are in $N^2 = \{0\}$, i.e. $s_1 s s_2 = 0 = s_1 s_2$. Thus, in all cases, $(t_1, [s, t], t_2)$ is the local identity of K_π at k, by Lemma 1.7. □

1.5. Rhodes expansion

In this section and the next ones, we turn to the description of a few expansions. The most classical, by which we shall start, is the *Rhodes expansion*. It has been extensively studied, in particular in [14, 6, 1, 3].

Let S be a semigroup. $\hat{S}^{\mathscr{R}}$ is the set of all finite strict $\mathscr{R}$-chains of elements of

S, i.e. the set of elements of the form $(s_1 >_{\mathscr{R}} \cdots >_{\mathscr{R}} s_n)$. $\hat{S}^{\mathscr{R}}$ is a semigroup with the product

$$(s_1 >_{\mathscr{R}} \cdots >_{\mathscr{R}} s_n)(t_1 >_{\mathscr{R}} \cdots >_{\mathscr{R}} t_m)$$
$$= \mathfrak{R}(s_1 >_{\mathscr{R}} \cdots >_{\mathscr{R}} s_n \geq_{\mathscr{R}} s_n t_1 \geq_{\mathscr{R}} \cdots \geq_{\mathscr{R}} s_n t_m)$$

where $\mathfrak{R}$ is the *reduction* given by $\mathfrak{R}(s) = (s)$, $\mathfrak{R}(s_1 \mathscr{R} s_2) = (s_2)$ and

$$\mathfrak{R}(s_1 \geq_{\mathscr{R}} \cdots \geq_{\mathscr{R}} s_n >_{\mathscr{R}} s_{n+1} \geq_{\mathscr{R}} \cdots \geq_{\mathscr{R}} s_{n+m})$$
$$= (\mathfrak{R}(s_1 \geq_{\mathscr{R}} \cdots \geq_{\mathscr{R}} s_n) >_{\mathscr{R}} \mathfrak{R}(s_{n+1} \geq_{\mathscr{R}} \cdots \geq_{\mathscr{R}} s_{n+m})).$$

$\eta : \hat{S}^{\mathscr{R}} \to S$, defined by $(s_1 >_{\mathscr{R}} \cdots >_{\mathscr{R}} s_n)\eta = s_n$ is an onto morphism.

The dual construction $\hat{S}^{\mathscr{L}}$ is the set of all finite strict $\mathscr{L}$-chains of elements of S $(s_n <_{\mathscr{L}} \cdots <_{\mathscr{L}} s_1)$, with a product defined dually, is also a semigroup, with canonical projection onto S $(s_n <_{\mathscr{L}} \cdots <_{\mathscr{L}} s_1)\eta = s_n$. Elementary manipulations of the products of $\hat{S}^{\mathscr{R}}$ and $\hat{S}^{\mathscr{L}}$ prove the following:

Lemma 1.10. (1) *Let* $s = (s_1 >_{\mathscr{R}} \cdots >_{\mathscr{R}} s_n)$ *and* $t = (t_1 >_{\mathscr{R}} \cdots >_{\mathscr{R}} t_m)$ *be in* $\hat{S}^{\mathscr{R}}$. *Then* $s \leq_{\mathscr{R}} t$ *(resp.* $s <_{\mathscr{R}} t$, $s \mathscr{R} t$*) iff* $n \geq m$ *(resp.* $n > m$, $n = m$*),* $s_m \mathscr{R} t_m$ *and* $s_i = t_i$ *for all* $1 \leq i < m$.

(2) *Let* $s = (s_n <_{\mathscr{L}} \cdots <_{\mathscr{L}} s_1)$ *and* $t = (t_1 <_{\mathscr{L}} \cdots <_{\mathscr{L}} t_1)$ *be in* $\hat{S}^{\mathscr{L}}$. *Then* $s \leq_{\mathscr{L}} t$ *(resp.* $s <_{\mathscr{L}} t$, $s \mathscr{L} t$*) iff* $n \geq m$ *(resp.* $n > m$, $n = m$*),* $s_m \mathscr{L} t_m$ *and* $s_i = t_i$ *for all* $1 \leq i < m$. □

The first author also proved the following in [6] for arbitrary semigroups:

Proposition 1.11. *For arbitrary semigroups, the Rhodes expansions have the following properties*:

(1) *Let* $s \in \hat{S}^{\mathscr{R}}$. *Its left stabilizer* $\mathrm{LStab}(s, \hat{S}^{\mathscr{R}}) = \{t \in \hat{S}^{\mathscr{R}} \mid ts = s\}$ *is* $\mathscr{L}$*-trivial bounded aperiodic and its regular elements are idempotent. Similarly, if* $s \in \hat{S}^{\mathscr{L}}$, *its right stabilizer* $\mathrm{RStab}(s, \hat{S}^{\mathscr{L}}) = \{t \in \hat{S}^{\mathscr{L}} \mid st = s\}$ *is* $\mathscr{R}$*-trivial bounded aperiodic and its regular elements are idempotent.*

(2) *If* J *is a null* $\mathscr{J}$*-class of* $\hat{S}^{\mathscr{R}}$ *or* $\hat{S}^{\mathscr{L}}$, *its Schützenberger group is trivial.* □

Finally, the first author proved the following proposition [6]. The proof we give here is different from the original one. One of the reasons [15] was written was to make this proof possible!

Proposition 1.12. *Let* S *be a semigroup and* M *be any non-trivial monoid. For any large enough integer* n, *there exists a division* $\theta : \hat{S}^{\mathscr{R}} < M^n * S$ *(resp.* $\theta : \hat{S}^{\mathscr{L}} < S *_{\mathrm{r}} M^n$*) such that* $\theta\pi = \eta$. *If* S *is a monoid,* θ *can be chosen to be a division* $\hat{S}^{\mathscr{R}} < M^n \circ S$ *(resp.* $\hat{S}^{\mathscr{L}} < S \circ_{\mathrm{r}} M^n$*).*

$$\begin{array}{ccc} M^n * S & \xrightarrow{\ \pi\ } & S \\ \theta \bigvee & \nearrow_{\eta} & \\ \hat{S}^{\mathscr{R}} & & \end{array} \qquad \begin{array}{ccc} S *_{\mathrm{r}} M^n & \xrightarrow{\ \pi\ } & T \\ \theta \bigvee & \nearrow_{\eta} & \\ \hat{S}^{\mathscr{L}} & & \end{array}$$

Proof. Since $\hat{S}^{\mathscr{L}} = (\widehat{S^{\mathrm{r}}}^{\mathscr{R}})^{\mathrm{r}}$, the statements relative to $\hat{S}^{\mathscr{L}}$ are a consequence of those relative to $\hat{S}^{\mathscr{R}}$. We prove the latter by using Propositions 1.3 and 1.1(3), and proving that $D_\eta \in \mathbf{II}$. Let $s \in S$ and $(s, [\hat{u}, u]) \in D_\eta(s)$, where $\hat{u} = (u_1 >_{\mathscr{R}} \cdots >_{\mathscr{R}} u_k = u)$ and $su = s$. We want to check that $(s, [\hat{u}, u])$ is the local identity of D_η at s. This is trivial (see the definition of D_η) as soon as we notice that for all $\hat{s} = (s_1 >_{\mathscr{R}} \cdots >_{\mathscr{R}} s_n = s) \in s\eta^{-1}$,

$$\begin{aligned}\hat{s}\hat{u} &= \Re(s_1 >_{\mathscr{R}} \cdots >_{\mathscr{R}} s_n = s \geq_{\mathscr{R}} su_1 \geq_{\mathscr{R}} \cdots \geq_{\mathscr{R}} su_k = su = s)\\ &= (s_1 >_{\mathscr{R}} \cdots >_{\mathscr{R}} su = s)\\ &= \hat{s}.\end{aligned}$$

□

1.6. Stiffler expansion

One simple variant of the Rhodes expansion is the Stiffler expansion of [11].

Let $\theta : S \to T$ be an onto morphism and let Q be a θ-singular $\mathscr{J}$-class of S. Let $\theta^I : S^I \to T^I$ be θ extended to I with $\theta^I(I) = I$ (and so $\forall s \in S$, $\theta^I(s) = \theta(s)$). For ease of notation we write $\theta^I \equiv \bar{\theta}$. Recall that Q^0 is isomorphic to some $\mathscr{M}^0(A, B, G, P)$, and let J' be the $\mathscr{J}$-class of T that contains $Q\theta$. Finally, let M be a non-trivial monoid, large enough so as to allow $M \setminus \{1\}$ to contain (set-theoretically) the set B. To every s of S, we associate an element $\bar{s}$ of $T^I \circ_{\mathrm{r}} M$, $\bar{s} = ({}_s f, s\theta)$ defined as follows: if $t_1 >_{\mathscr{J}} J'$, in particular $t_1 \bar{\theta}^{-1} = \{s_1\}$ for some $s_1 \in S^I \setminus Q$: if $ss_1 \in L_b^S \subseteq Q$, then ${}_s f(t_1) = b$; in all other cases ${}_s f(t_1) = 1$. The (right) *Stiffler expansion* ${}_Q T$ is defined to be the subsemigroup of $T^I \circ_{\mathrm{r}} M$ generated by the $\bar{s}$ $(s \in S)$. We denote by η the morphism from ${}_Q T$ onto T that maps $\bar{s}_n \cdots \bar{s}_1$ to $\theta(s_n) \cdots \theta(s_1)$.

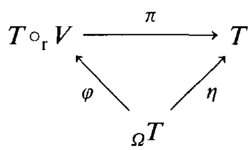
Let $s_n, \ldots, s_1$ be in S.

$$\bar{s}_n \cdots \bar{s}_1 = ({}_{s_n} f, s_n \theta) \cdots ({}_{s_1} f, s_1 \theta) = (g, (s_n \cdots s_1)\theta).$$

For each t in T,

$$g(t) = {}_{s_n} f((s_{n-1} \cdots s_1)\theta t) \cdots {}_{s_2} f((s_1)\theta t) {}_{s_1} f(t).$$

At most one index i $(1 \leq i \leq n)$ satisfies $(s_{i-1} \cdots s_1)\bar{\theta} t >_{\mathscr{J}} J'$ and $(s_i \cdots s_1)\bar{\theta} t \in J'$ (with the convention that $\emptyset = I$). So $g(t_1) \in B \cup \{1\} \subseteq M$. Consequently, ${}_Q T$ is independent of the choice of M and hence, for every non-trivial M-variety $\mathbf{V}$ and every $V \in \mathbf{V}$ that is large enough, there is an injective morphism $\varphi : {}_Q T \to T \circ_{\mathrm{r}} V$ such that $\varphi\pi = \eta$.



The idea behind the construction of ${}_Q T$ can be heuristically described as follows. The element $\bar{s}_n \cdots \bar{s}_1 = (g, (s_n \cdots s_1)\theta) \in {}_Q T$ records the value of $(s_n \cdots s_1)\theta$ and for each $\mathscr{L}$-chain $s_n \cdots s_1 s \leq_{\mathscr{L}} \cdots \leq_{\mathscr{L}} s_1 s \leq_{\mathscr{L}} s$ $(s \in S)$ the $\mathscr{L}$-class of Q (unique if it exists and equals $g(\theta(s))$) which contains some element of the chain $g(\theta(s)) = 1$ when such an $\mathscr{L}$-class does not exist. Otherwise said, one 'remembers' the B coordinate of the first fall into Q from the right going left.

We can also consider the dual construction T_Q.

1.7. Karnofsky expansion

An extension of the Rhodes expansion is the Karnofsky expansion. Let $\theta: S \to T$ be an onto morphism. Let S^0 be the semigroup obtained by adjoining a zero to S, and S' be the set of all $(s,t) \in S^0 \times T^I$ such that $s\theta t <_{\mathscr{L}} t$. The *Karnofsky expansion* $\hat{T}_\theta$ is the set of all elements $a = [(0,t_n)(s_{n-1},t_{n-1}) \cdots (s_1,t_1)]$ of $(S')^n$ $(n \geq 2)$ such that $s_i \in S$ $(1 \leq i \leq n-1)$, $t_i \in T$ $(2 \leq i \leq n)$, $t_1 = I$ and $t_{i+1} \mathscr{L} s_i \theta t_i <_{\mathscr{L}} t_i$. Note that $(t_n <_{\mathscr{L}} \cdots <_{\mathscr{L}} t_2 <_{\mathscr{L}} I) \in \widehat{T^I}^{\mathscr{L}}$.

We define in $\hat{T}_\theta$ a product inspired by the one in $\widehat{T^I}^{\mathscr{L}}$. The product $[(0,t_n) \cdot (s_{n-1},t_{n-1}) \cdots (s_2,t_1)(s_1,I)] \cdot [(0,t'_m)(s'_{m-1},t'_{m-1}) \cdots (s'_2,t'_2)(s'_1,I)]$ is equal to $[(0,t_n t'_m) \cdot (s_{i_k}, t_{i_k} t'_m) \cdots (s_{i_1}, t_{i_1} t'_m)(s'_{m-1},t'_{m-1}) \cdots (s'_2,t'_2)(s'_1,I)]$ if $1 \leq i_1 < \cdots < i_k < n$ are such that

$$(t_n <_{\mathscr{L}} \cdots <_{\mathscr{L}} t_1)(t'_m <_{\mathscr{L}} \cdots <_{\mathscr{L}} t'_1)$$
$$= (t_n t'_m <_{\mathscr{L}} t_{i_k} t'_m <_{\mathscr{L}} \cdots <_{\mathscr{L}} t_{i_1} t'_m <_{\mathscr{L}} t'_{m-1} <_{\mathscr{L}} \cdots <_{\mathscr{L}} t'_2 <_{\mathscr{L}} I).$$

It is easy to check that $\hat{T}_\theta$ is a semigroup, that $\pi: \hat{T}_\theta \to T$ is an onto morphism and that $\hat{T}_\theta$ is generated by the elements of the form $[s] = [(0,s\theta)(s,I)]$ $(s \in S)$.

Heuristically, the product $[s_n] \cdots [s_1]$ $(s_1, \ldots, s_n \in S)$ records the strict $\mathscr{L}$-chain given by taking the reduction of $(s_n \cdots s_1)\theta \leq_{\mathscr{L}} \cdots \leq_{\mathscr{L}} s_1\theta < I$ together with, for each leap from one $\mathscr{L}$-class of T to the next, the particular s_i that triggered it 'at the beginning'.

Let us finally notice that the mapping $\eta: \hat{T}_\theta \to \hat{T}^{\mathscr{L}}$ defined by

$$[s_n] \cdots [s_1]\eta = \Re((s_n \cdots s_1)\theta \leq_{\mathscr{L}} \cdots \leq_{\mathscr{L}} s_1\theta)$$

is an onto morphism.

Proposition 1.13. *For any large enough monoid M, there exists a division $\varphi: \hat{T}_\theta < T \circ_{\mathrm{r}} M$ such that $\varphi\pi = \eta$.*

Proof. By the dual of Proposition 1.3 and 1.12 and by Proposition 1.1(3) it suffices to show for $D \equiv (D_{\eta^{\mathrm{r}}})^{\mathrm{r}}$ (so D is the dual of D_η) that $D(\hat{t}) \subseteq \{1\}$ for all $\hat{t} \in \hat{T}_\theta \cdot (s_n < \cdots < s_1 < I) \in \widehat{T^I}^{\mathscr{L}}$ and let $\alpha \in \hat{T}_\theta$, $\eta(\alpha) = (t_m < \cdots < t_1)$. Then $\eta(\alpha) \cdot (s_n < \cdots < s_1 < I) = (s_n < \cdots < s_1 < I)$ iff $t_m s_n = s_n$. But then directly from the definition of multiplication in $\hat{T}_\theta$, if $\beta \in \hat{T}_\theta$ and $\eta(\beta) = (s_n < \cdots < s_1 < I)$, then $\alpha\beta = \beta$. $\square$

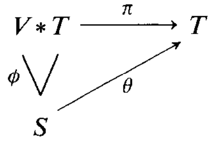
2. Decomposition of m.p.s.'s

We are now ready to decompose the m.p.s.'s. Recall that we had split the m.p.s.'s into four classes. We shall examine each one in turn, using the notation of [10, Section 3].

2.1. Class I

Let $\theta : S \to T$ be an m.p.s. of class I. Recall that there exists a unique θ-singular $\mathcal{J}$-class J, that $J' = J\theta$ is a $\mathcal{J}$-class, that if G and G' are the Schützenberger groups respectively of J and J', then $G\theta = G'$, and that N denotes the group $\ker(\theta)$. Further, N is a minimal non-trivial normal subgroup of G and hence $N \cong \bar{S} \times \cdots \times \bar{S}$ with $\bar{S}$ a finite simple group, so $(N) = (\bar{S})$.

Proposition 2.1. *There exists a group V in the* **G***-variety $(N) = (\bar{S})$ with $\bar{S}$ a finite simple group and a division $\varphi : S < V * T$ such that $\varphi\pi = \theta$. Further, if θ is a morphism of monoids, we may choose V so that $\varphi : S < V \circ T$.*

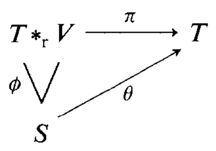


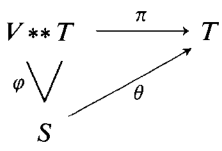
Proof. By Proposition 1.3, it is enough to show that $D_\theta \in g(N)$, which is equivalent, after Proposition 1.1(4), to showing that the non-empty base semigroups of D_θ are groups in (N), whose unit is the local identity. This basically follows from Green's lemma and its application to the Schützenberger group. The details go as follows.

Let $t_1 \in T^I$ be an object of D_θ. If $t_1 \notin J'$, then $t_1\theta^{-1}$ has at most one element, so that every map from $t_1\theta^{-1}$ into itself is the identity: every element of $D_\theta(t_1)$ is the local identity.

Recall that, with the notations of [10, Subsection 3.2], $J^0 = \mathcal{M}^0(A, B, G, P)$ and $J'^0 = \mathcal{M}^0(A, B, G/N, P/N)$. If $t_1 \in J'$, then $t_1 = (a, g_1N, b)$ for some $a \in A$, $b \in B$, $g_1 \in G$, and $t_1\theta^{-1} = \{a\} \times g_1N \times \{b\}$. Let $(t_1, [s, t]) \in D_\theta(t_1)$ ($t = s\theta$ and $t_1t = t_1$). For each $s_1 = (a, g, b)$ ($g \in g_1N$) in $t_1\theta^{-1}$, s_1s is also in $t_1\theta^{-1}$ (since $(s_1s)\theta = t_1t = t_1$). Furthermore, $s_1s = (u_a(h_0 \cdot g)v_b)s = u_a((h_0 \cdot g)v_bs\bar{v}_b)v_b$. Let $\varrho_{b,s}$ be the right translation of $H^S_{1,1}$ with factor $v_bs\bar{v}_b$: $\varrho_{b,s} \in G$ and $s_1s = (a, g\varrho_{b,s}, b)$. Also, the mapping $\varrho : D_\theta(t_1) \to G$ that maps $(t_1, [s, t])$ to $\varrho_{b,s}$ is well defined and a morphism. Further, the above calculation of s_1s ($s_1 \in t_1\theta^{-1}$) shows that $(g\varrho_{b,s})N = gN$ for all $g \in g_1N$, so that $\varrho_{b,s} \in N$. Finally, the morphism ϱ is one-to-one. Indeed, if $(t_1, [s, t])$ and $(t_1, [s', t'])$ are in $D_\theta(t_1)$, and $\varrho_{b,s} = \varrho_{b,s'}$, the corresponding translations, from $t_1\theta^{-1}$ into itself, coincide and hence $(t_1, [s, t]) = (t_1, [s', t'])$. Thus, $D_\theta(t_1)$ is either empty or a subgroup of N with unit the local identity. □

Corollary 2.2. *The same result holds if we replace $V * T$ and $V \circ T$ by $T *_r V$ and $T \circ_r V$ (resp. $V ** T$ and $V \circ\circ T$).*

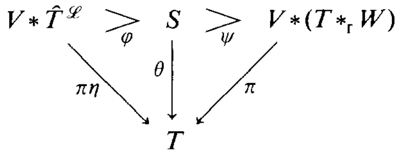




Proof. We know that if θ is in class I, then θ^{r} is too. So $(D_\theta)^{\mathrm{r}} = D_{\theta^{\mathrm{r}}}$ is in $g(N)$ and hence D_θ is in $g(N^{\mathrm{r}})$, which is equal to $g(N)$ since N is a group. Also, we know that $K_\theta < D_\theta$ (see Proposition 1.5). We conclude by using Proposition 1.3. $\square$

If θ is in class I_R, the above results are optimal. Indeed, J is regular and we may assume that $H^S_{1,1}$ is a group. Then $H^S_{1,1} = G$ and $H^T_{1,1} = H^S_{1,1}\theta = G/N$. If $\psi : S < V * T$ (resp. $S < V ** T$), then $G = H^S_{1,1} < H^S_{1,1}\psi \subseteq V * H^S_{1,1}\theta = V * G/N$ (resp. $G < V ** G/N$). After Propositions 1.3 and 1.4, N must divide V. On the other hand, it is natural to think that if θ is in class I_N, we shall not need groups to decompose it, since θ is then aperiodic (see [10, Proposition 3.7]). Indeed, we have

Proposition 2.3. *Let $\theta : S \to T$ be in class I_N, and $\mathbf{V}$ and $\mathbf{W}$ be non-trivial M-varieties. Denote by η the projection $\eta : \hat{T}^{\mathscr{L}} \to T$ (resp. $\eta : \hat{T}^{\mathscr{R}} \to T$). Then there exist $V \in \mathbf{V}$, $W \in \mathbf{W}$, $\varphi : S < V * \hat{T}^{\mathscr{L}}$ (resp. $S < \hat{T}^{\mathscr{R}} *_{\mathrm{r}} V$ and $\psi : S < V * (T *_{\mathrm{r}} W)$ (resp. $S < (W * T) *_{\mathrm{r}} V$) such that $\varphi\pi\eta = \psi\eta = \theta$.*



Proof. The result relative to $\hat{T}^{\mathscr{R}} *_{\mathrm{r}} V$ is a consequence of the one relative to $V * \hat{T}^{\mathscr{L}}$ and of the fact that θ^{r} is also in class I_N. Let $\sigma = \theta\eta^{-1} : S \to \hat{T}^{\mathscr{L}}$ be the relational morphism lifting θ. It is enough to show that the non-empty base semigroups of D_φ contain only the local identity: then, by Proposition 1.1(3) and 1.3, we shall have $\psi : S < V * \hat{T}^{\mathscr{L}}$ for some V in $\mathbf{V}$ such that $\psi\pi = \varphi$. Proposition 1.12 will then allow us to finish the proof.

Let then $\hat{t}_1 \in (\hat{T}^{\mathscr{L}})^I$ be an object of D_φ. If $\hat{t}_1 = I$, then $D_\varphi(\hat{t}_1) = \emptyset$. Otherwise, $\hat{t}_1 = (t_1 = y_n <_{\mathscr{L}} \cdots <_{\mathscr{L}} y_1)$. If $t_1 \notin J'$, $\hat{t}_1\varphi^{-1}$ has one element and hence, if $D_\varphi(\hat{t}_1) \neq \emptyset$, $D_\varphi(\hat{t}_1)$ contains only the local identity.

So let us assume that $t_1 = \hat{t}_1\eta \in J' = J\theta$. Since θ is in class I_N, J and J' are null $\mathscr{J}$-classes and both t_1 and $\hat{t}_1$ are null. Let $\hat{t} \in \hat{T}^{\mathscr{L}}$ be such that $\hat{t}_1\hat{t} = \hat{t}_1$. Then, after Lemma 1.10(2), $\hat{t}_1 \leq_{\mathscr{L}} \hat{t}$ and hence $\hat{t}$ is equal to $(t <_{\mathscr{L}} y_{k-1} <_{\mathscr{L}} \cdots <_{\mathscr{L}} y_1)$, with $k \leq n$ and $t\mathscr{L}y_k$. Let now $s \in \hat{t}\varphi^{-1} = t\theta^{-1}$ and let us denote by ϱ_s the right translation of $\hat{t}_1\varphi^{-1}$ by s: ϱ_s maps $\hat{t}_1\varphi^{-1}$ into itself. So there exists an integer n_0 such that, for all $n \geq n_0$, $(\varrho_s)^n = \varrho_{s^n}$ is a regular element of the monoid of all mappings from $\hat{t}_1\varphi^{-1}$ into itself, and s^n, t^n and $\hat{t}^n$ are regular elements, respectively of $T_{t_1}\theta^{-1}$, T_{t_1} and $(\hat{T}^{\mathscr{L}})_{\hat{t}_1}$. If ϱ_s is not the identity on $\hat{t}_1\varphi^{-1}$, there exists a power $n \geq n_0$ of ϱ_s which is regular and not the identity either. So we may choose in $D_\varphi(\hat{t}_1)$ a regular element $(\hat{t}_1, [s, \hat{t}])$ with $\hat{t}$ regular in $(\hat{T}^{\mathscr{L}})_{\hat{t}_1}$ and ϱ_s is not the identity on $\hat{t}_1\varphi^{-1}$.

After Proposition 1.11(1), $\hat{t}$ is then idempotent and hence $t^2 = t$ in T. Since t_1 is null, we cannot have $t\mathscr{L}t_1$, so that $\hat{t} = (t <_{\mathscr{L}} y_{k-1} <_{\mathscr{L}} \cdots <_{\mathscr{L}} y_1)$ with $k < n$. Also, $t \notin J'$ since J' is null. Further, for all $1 \leq i < n$, $y_i >_{\mathscr{L}} y_n = t_1$ and hence $y_i \notin J'$. For all

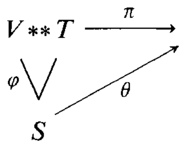
$1 \le i < n$ we shall denote by z_i (resp. e) the unique element of $y_i\theta^{-1}$ (resp. $t\theta^{-1}$). Then, necessarily, $s = e = e^2$.

Let now $s_1 \in \hat{t}_1\varphi^{-1} = t_1\theta^{-1}$. For all $1 \le i < n$, $s_1 <_{\mathscr{L}} z_i$ in S. Indeed, $t_1 <_{\mathscr{L}} y_i$: there exists u in T such that $t_1 = uy_i$. If $\bar{u} \in u\theta^{-1}$, $(\bar{u}z_i)\theta = t_1$. Since θ is an $\mathscr{H}$-morphism, $\bar{u}z_i \mathscr{L} s_1$ so that $s_1 \le_{\mathscr{L}} z_i$. Furthermore, $s_1 \in J$, $z_i \notin J$, and hence $s_1 <_{\mathscr{L}} z_i$. In particular, $s_1 <_{\mathscr{L}} z_k \mathscr{L} e = s$ and hence $s_1 = s_1 e = s_1 s$. But this makes ϱ_s the identity, in contradiction with our assumption.

Consequently, $D_\varphi(\hat{t}_1)$ has at most one element, which corresponds to the identity of $\hat{t}_1\varphi^{-1}$ and hence is the local identity. □

Finally, still in the case where θ is in class I_N, we have

Proposition 2.4. *If $\mathbf{V}$ is a non-trivial M-variety, there exists V in $\mathbf{V}$ and a division $\varphi : S < V ** T$ such that $\varphi p = \theta$. If θ is a monoid morphism, we may choose $\varphi : S < V \circ\circ T$.*



Proof. After Propositions 1.8 and 1.1(3), it is enough to show that the non-empty base semigroups of K_θ contains at most one element, which is the local identity.

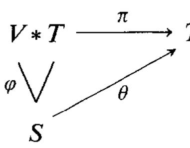
So, let $d = (t_1, t_2) \in T^I \times T^I$ be an object of K_θ and let $(t_1, [s, t], t_2) \in K_\theta(d)$: $t_1 t = t_1$, $t_2 = tt_2$ and $t = s\theta$. Let also $s_1 \in t_1\theta^{-1}$ and $s_2 \in t_2\theta^{-1}$: $(s_1 s s_2)\theta = t_1 t_2 = (s_1 s_2)\theta$. If $t_1 \notin J'$, $s_1 s = s_1$ since θ is one-to-one on $S \setminus J = S \setminus J'\theta^{-1}$, and hence $s_1 s s_2 = s_1 s_2$. Similarly, if $t_2 \notin J'$, $ss_2 = s_2$ and hence $s_1 s s_2 = s_1 s_2$. Finally, if t_1 and t_2 are in J', then $t_1 t_2 \notin J'$, since J' is null, so here too $s_1 s s_2 = s_1 s_2$. We may now finish, thanks to Lemma 1.7. □

Remark. A similar proof shows that the same result holds if θ is in class $\mathrm{III}_{N>N}$ or IV.

2.2. Class II

Let $\theta : S \to T$ be in class II. We shall use the notations of [10, Subsection 3.3].

Proposition 2.5. *If θ identifies rows and $\mathbf{V}$ is a non-trivial M-variety, there exist V in $\mathbf{V}$ and a division $\varphi : S < V * T$ such that $\varphi\pi = \theta$. If θ is a monoid morphism, we may choose $\varphi : S < V \circ T$.*

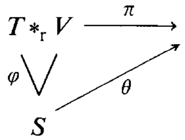


Proof. After Propositions 1.3 and 1.1(3), it is enough to show that the non-empty base semigroups of D_θ contain exactly one element, which is the local identity.

So let $t_1 \in T^I$ be an object of D_θ. If $t_1 \notin J'$, $t_1\theta^{-1}$ has only one element, so that any element of $D_\theta(t_1)$ corresponds to the identity on $t_1\theta^{-1}$ and hence is the local identity. If $t_1 \in J'$, $t_1 = (a', g, b)$ for some $a' \in A'$, $g \in G$, $b \in B$ and $t_1\theta^{-1} = a'\alpha^{-1} \times \{g\} \times \{b\}$. Let $(t_1, [s,t]) \in D_\theta(t_1)$: $t_1 t = t_1$. So, for all $a_1 \in a'\alpha^{-1}$, $(a_1, g, b)s = (a_2, g, b)$ for some $a_2 \in a'\alpha^{-1}$. Thus $(a_2, g, b)\mathscr{L}(a_1, g, b)$ and $(a_2, g, b) \leq_{\mathscr{R}} (a_1, g, b)$, and hence $(a_2, g, b)\mathscr{H}(a_1, g, b)$, i.e. $a_1 = a_2$. So $(a_1, g, b)s = (a_1, g, b)$, $(t_1, [s,t])$ induces the identity on $t_1\theta^{-1}$ and $(t_1, [s,t])$ is the local identity. □

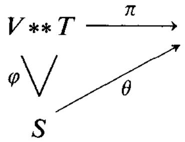
If θ identifies columns, by [10, Proposition 3.5], θ^{r} identifies rows. By demonstrations dual to the ones of Corollary 2.2, we obtain the following corollary:

Corollary 2.6. *If θ identifies columns and $\boldsymbol{V}$ is a non-trivial $\boldsymbol{M}$-variety, there exist V in $\boldsymbol{V}$ and a division $\varphi : S < T *_{\mathrm{r}} V$ such that $\varphi\pi = \theta$.* If *θ is a monoid morphism, we may choose $\varphi : S < T \circ_{\mathrm{r}} V$.*



□

Corollary 2.7. *If θ is in class* II *and $\boldsymbol{V}$ is a non-trivial $\boldsymbol{M}$-variety, there exist V in $\boldsymbol{V}$ and a division $\varphi : S < V ** T$ such that $\varphi\pi = \theta$. If θ is a monoid morphism, we may choose $\varphi : S < V \circ\circ T$.*



Proof. Use Corollary 2.6 and Propositions 1.5 and 1.8. □

2.3. Class III

Let $\theta : S \to T$ be in class III. Recall that we denote by U_1 the monoid $\{0, 1\}$ that generates $\boldsymbol{J}_1$ and lies in every $\boldsymbol{M}$-variety that is not a $\boldsymbol{G}$-variety. The somewhat long proof that follows uses Stiffler expansion ${}_{Q}T$ and Karnofsky expansion $\hat{T}_\theta$ (see Subsections 1.6 and 1.7).

Proposition 2.8. *Let θ be an m.p.s. of class* III *and let $\boldsymbol{V}$ be a non-trivial $\boldsymbol{M}$-variety. Then there exist $X \in \boldsymbol{J}_1$, $V \in \boldsymbol{V}$ and a division $\varphi : S < X * (T *_{\mathrm{r}} V)$ (resp. $S < (V * T) *_{\mathrm{r}} X$) such that $\varphi\pi = \theta$.*

If θ is a monoid morphism, we may choose $\varphi : S < X \circ (T \circ_{\mathrm{r}} V)$, etc...

Proof. We know that, for all large enough W_1 and W_2 in $\mathbf{W}$, ${}_QT \subseteq T \circ_r W_1$ and $\hat{T}_\theta < T \circ_r W_2$. Let Z be the subsemigroup of $\hat{T}_\theta \times {}_QT$ generated by the elements $([(0, s\theta)(s, I)], \bar{s})$ $(s \in S)$. A canonical projection π is still defined from Z onto T, that maps $([0, s\theta)(s, I)], \bar{s})$ onto $s\theta$, and one can check easily that Z divides $T \circ_r (W_1 \times W_2)$. So, for all large enough W in $\mathbf{W}$, $Z < T \circ_r W$.

We shall (a) construct an onto relational morphism $\varphi : Z \to S$ and (b) prove that the non-empty base semigroups of $D_{\varphi^{-1}}$ are idempotent commutative semigroups. This will prove the proposition, using Propositions 1.3 and 1.1.

(a) For all s in S, we denote by $[s]$ the element $([(0, s\theta)(s, I)], \bar{s})$ of Z. Given $s_n, \dots, s_1$ in S, we define $([s_n] \cdots [s_1])\varphi$ as follows:

Case 1. If $(s_n \cdots s_1)\theta \notin J'$, $([s_n] \cdots [s_1])\varphi = \{s_n \cdots s_1\}$;

Case 2. If $(s_n \cdots s_1)\theta \in J'$ and for all $1 \le i \le n$, $s_i \cdots s_1 \notin Q$, then again $([s_n] \cdots [s_1])\varphi = \{s_n \cdots s_1\}$. In this case, if $(f, (s_n \cdots s_1)\theta) = \bar{s}_n \cdots \bar{s}_1$, then $f(1) = 1$; in particular, $s_n \cdots s_1 \in J$;

Case 3. Finally, if $(s_n \cdots s_1)\theta \in J'$ and $s_{i_0} \cdots s_1 \in L_b \subseteq Q$ for some $1 \le i_0 \le n$ and $b \in B$, then

$$([s_n] \cdots [s_1])\varphi = ((s_n \cdots s_1)\theta\theta^{-1} \cap J) \cup ((s_n \cdots s_1)\theta\theta^{-1} \cap L_b).$$

Note that i_0 is not necessarily unique, but that b is: if $(f, (s_n \cdots s_1)\theta = \bar{s}_n \cdots \bar{s}_1$, then $f(1) = b$. Note also that $(s_n \cdots s_1)\theta\theta^{-1} \cap J$ contains exactly one element, but that $(s_n \cdots s_1)\theta\theta^{-1} \cap L_b$ may be empty.

φ is well defined. Indeed, if $[s_n] \cdots [s_1] = [s'_m] \cdots [s'_1]$, then $(s_n \cdots s_1)\theta = ([s_n] \cdots [s_1])\pi = ([s'_m] \cdots [s'_1])\pi = (s'_m \cdots s'_1)\theta$ and $\bar{s}_n \cdots \bar{s}_1 = \bar{s}'_m \cdots \bar{s}'_1$. By construction of ${}_QT$, this last equality implies that there exists $1 \le i_0 \le n$ such that $s_{i_0} \cdots s_1 \in L_b$ iff there exists $1 \le j_0 \le m$ such that $s'_{j_0} \cdots s'_1 \in L_b$. Let us note also that $s_n \cdots s_1$ always lies in $([s_n] \cdots [s_1])\varphi$ and that $([s_n] \cdots [s_1])\varphi \subseteq (s_n \cdots s_1)\theta\theta^{-1}$. So φ is onto. Note that, if $(s_n \cdots s_1)\theta$ is in J', then $(s_n \cdots s_1)\theta\theta^{-1} \cap J$ has exactly one element.

We now turn to showing that φ is a relational morphism. Let $s_n, \dots, s_1$, $s'_m, \dots, s'_1$ be in S. If $[s_n] \cdots [s_1]$ and $[s'_m] \cdots [s'_1]$ are as in Case 1 or 2, then $([s_n] \cdots [s_1])\varphi([s'_m] \cdots [s'_1])\varphi$ consists only of $s_n \cdots s_1 s'_m \cdots s'_1$ and hence lies in $([s_n] \cdots [s_1][s'_m] \cdots [s'_1])\varphi$.

If $[s_n] \cdots [s_1]$ and $[s'_m] \cdots [s'_1]$ are both in Case 3, with b and b' the associated elements of B, let x and y be respectively in $([s_n] \cdots [s_1])\varphi$ and $([s'_m] \cdots]s'_1])\varphi$. Then $x\theta = (s_n \cdots s_1)\theta$ and $y\theta = (s'_m \cdots s'_1)\theta$ are in J'. Since θ is one-to-one on $S \setminus Q$, if $xy \notin Q$, then $(xy)\theta = (s_n \cdots s_1 s'_m \cdots s'_1)\theta$ implies that xy lies in $([s_n] \cdots [s_1][s'_m] \cdots [s'_1])\varphi$. This is the case in particular if at least one of x and y is in J. Suppose on the contrary that $x \in L_b$ and $y \in L_{b'}$ and $xy \in Q$. Then $xy \mathscr{L} y$, so that $xy \in L_{b'}$ and hence $xy \in ([s_n] \cdots [s_1][s'_m] \cdots [s'_1])\varphi$.

If $[s_n] \cdots [s_1]$ is as in Case 1 or 2 and $[s'_m] \cdots [s'_1]$ as in Case 3, with b' the associated element of B, let $y \in ([s'_m] \cdots [s'_1])\varphi$. If $s_n \cdots s_1 y \notin Q$, then, as above, using the injective of θ on $S \setminus Q$, we obtain $s_n \cdots s_1 y \in ([s_n] \cdots [s_1][s'_m] \cdots [s'_1])\varphi$. If $s_n \cdots s_1 y \in Q$, then $y \in L_{b'}$ and $s_n \cdots s_1 y \le_{\mathscr{L}} y$, so that $s_n \cdots s_1 y \in L_{b'}$. Again this shows that $s_n \cdots s_1 y \in ([s_n] \cdots [s_1][s'_m] \cdots [s'_1])\varphi$.

The last case occurs when $[s_n]\cdots[s_1]$ is as in Case 3 (with b the associated element of B) and $[s'_m]\cdots[s'_1]$ is as in Case 1 or 2. Let $x\in([s_n]\cdots[s_1])\varphi$. As above, if $xs'_m\cdots s'_1\notin q$, then $xs'_m\cdots s'_1\in([s_n]\cdots[s_1][s'_m]\cdots[s'_1])\varphi$. Otherwise, $xs'_m\cdots s'_1\in Q$. But $x\mathscr{L}s_{i_0}\cdots s_1$ for some $1\le i_0\le n$, so that $xs'_m\cdots s'_1$ is $\mathscr{L}$-equivalent to $s_{i_0}\cdots s_1 s'_m\cdots s'_1$, thus proving that $xs'_m\cdots s'_1\in([s_n]\cdots[s_1][s'_m]\cdots[s'_1])\varphi$.

(b) Since φ is a relational morphism from Z onto S, $\varphi^{-1}:S\to Z$ is also an onto relational morphism. We want to show that the non-empty base semigroups of $D_{\varphi^{-1}}$ are idempotent and commutative. Let $s_n,\dots,s_1\in S$ and $z=[s_n]\cdots[s_1]\in Z$ be such that $D_{\varphi^{-1}}(z)\ne\emptyset$. The set associated to z is $z\varphi=([s_n]\cdots[s_1])\varphi$. If $z\varphi$ has only one element, then the identity is the only mapping from $z\varphi$ into itself, and hence $D_{\varphi^{-1}}(z)$ contains only the local identity.

Let us suppose, otherwise, that $(s_n\cdots s_1)\theta\in J'$, that $s_{i_0}\cdots s_1\in L_b\subseteq Q$ for some $1\le i_0\le n$ and that $(s_n\cdots s_1)\theta\theta^{-1}\cap L_b\ne\emptyset$. Let $\hat{s}$ be the unique element of $(s_n\cdots s_1)\theta\theta^{-1}\cap J=z\varphi\cap J$. Let also $s'_1,\dots,s'_p\in S$, $z'=[s'_p]\cdots[s'_1]\in Z$ and $s'\in z'\varphi$ be such that $zz'=z$. In particular, $s'\theta=(s'_p\cdots s'_1)\theta$ and $(s_n\cdots s_1 s'_p\cdots s'_1)\theta=(s_n\cdots s_1)\theta$. Let ϱ be the right translation of $z\varphi$ by s'.

If $s'\in J$, then for each $x\in z\varphi$, we have $xs'\in J$ and hence $xs'=\hat{s}$. So ϱ is the constant function $\hat{s}$.

If $s'\in Q$, then $s'\in L_{b'}$ for some $b'\in B$, and there exist $1\le p_0\le p$ such that $s'_{p_0}\cdots s'_1\in L_{b'}$. But $zz'=z$ implies that $b=b'$, so that $s'\in L_b$. Let $x\in z\varphi$. If $x=\hat{s}$, then $xs'=x$. If $x\in z\varphi\cap L_b$, then xs' lies in Q iff s' is $\mathscr{H}$-equivalent to some idempotent, so that $xs'\in Q$ for all the x's or for none of the x's in $z\varphi\cap L_b$. If $xs'\notin Q$, then $xs'=\hat{s}$. If $xs'Q$, then $xs'\mathscr{H}x$, and hence $xs'=x$ since θ is injective on $\mathscr{H}$-classes. So ϱ is either the identity or the constant function on $\hat{s}$.

Finally, let us consider the case where $s'\theta\notin J'$, so that $s'\theta>_{\mathscr{J}}J'$. In this case, we prove that ϱ is necessarily the identity. Assume indeed that ϱ is not the identity. As we did in the proof of Proposition 2.3, we can assume that z' is a regular element of Z_z, the right stabilizer of z in Z. Recall that the canonical mapping $\eta:\hat{T}_\theta\to\hat{T}^{\mathscr{L}}$ (see Subsection 1.7) is an onto morphism. So, in $\hat{T}^{\mathscr{L}}$, $\mathfrak{R}((s'_p\cdots s'_1)\theta\le_{\mathscr{L}}\cdots\le_{\mathscr{L}}s'_1\theta)$ is in the right stabilizer of $\mathfrak{R}((s_n\cdots s_1)\theta\le_{\mathscr{L}}\cdots\le_{\mathscr{L}}s_1\theta)$ and hence there exists $k\le n$ such that

$$(s'_p\cdots s'_1)\theta\mathscr{L}(s_{k+1}\cdots s_1)\theta,$$

$$\mathfrak{R}((s'_p\cdots s'_1)\theta\le_{\mathscr{L}}\cdots\le_{\mathscr{L}}s'_1\theta)=\mathfrak{R}((s'_p\cdots s'_1)\theta\le_{\mathscr{L}}(s_k\cdots s_1)\theta\le_{\mathscr{L}}\cdots\le_{\mathscr{L}}s_1\theta).$$

Since $s'\theta>_{\mathscr{J}}J'$, we have $k+1<i_0\le n$ and $s'=s'\theta\theta^{-1}$. Furthermore, z' is regular, so the η-image of its $\hat{T}_\theta$-component, $\hat{s}'=\mathfrak{R}((s'_p\cdots s'_1)\theta\le_{\mathscr{L}}\cdots\le_{\mathscr{L}}s'_1\theta)$, is regular in the right stabilizer of the element $\mathfrak{R}((s_n\cdots s_1)\theta\le_{\mathscr{L}}\cdots\le_{\mathscr{L}}s_1\theta)$ of $\hat{T}_\theta$. After Proposition 1.11(1), $\hat{s}'$ is idempotent, and hence so is its projection in T, $(s'_p\cdots s'_1)\theta$. In particular, $s'=s'_p\cdots s'_1$ is an idempotent of S. Let then $x\in z\varphi$. If $x=\hat{s}$, then $xs'=x$. Otherwise, $x\in Q$. Then

$$x\mathscr{L}s_{i_0}\cdots s_1<_{\mathscr{L}}s_{k+1}\cdots s_1\mathscr{L}s'$$

and hence $xs'=x$. So ϱ fixes every element of $z\varphi$, in contradiction with our hypothesis. Thus ϱ is the identity on $z\varphi$.

So in every case, $D_{\varphi^{-1}}$ is a subsemigroup of U_1, and hence is idempotent and commutative. □

Corollary 2.9. *Let θ be an m.p.s. of class* III *and let V be a non-trivial M-variety. Then there exist $X \in J_1$, $V \in V$ and a division $\varphi : S < X ** (T *_r V)$ (resp. $S < (V * T) ** X$) such that $\varphi\pi = \theta$.*

If θ is a monoid morphism, we may choose $\varphi : S < X \circ\circ (T \circ_r V)$, etc...

Proof. This is a consequence of Proposition 2.8 and of the fact that K_θ divides D_θ (see Proposition 1.5). □

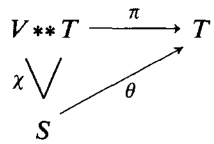
If θ is in class $\mathrm{III}_{R>R}$, the above results are optimal in a sense. In this case indeed, J', J and Q are regular $\mathcal{J}$-classes and there exist idempotents e, e_1 and e_0, respectively in J', Q and J such that $e\theta^{-1} = \{e_1, e_0\}$ is isomorphic to U_1: so, under θ, a subsemigroup isomorphic to U_1 has 'vanished'. More formally, we have

Corollary 2.10. *If θ is in class $\mathrm{III}_{R>R}$ then $U_1 < D_\theta$ and $U_1 < K_\theta$.*

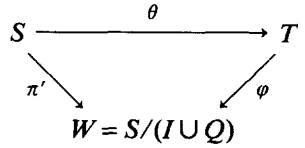
Proof. We show that $U_1 < D_\theta$. The result concerning K_θ is proved similarly. Since Q, J and J' are regular $\mathcal{J}$-classes, we can choose an idempotent e_1 in Q. Then $e = e_1\theta \in J'$ is idempotent and $e\theta^{-1} = \{e_1, e_0\}$ for some idempotent e_0 in J. Since $e_1 e_0 \in e\theta^{-1}$ and $e_1 e_0 \leq_{\mathcal{J}} e_0 <_{\mathcal{J}} e_1$, $e_1 e_0 = e_0$, so that $e\theta^{-1}$ is isomorphic to U_1. Let us then consider $D_\theta(e)$: $a_0 = (e, [e_0, e])$ and $a_1 = (e, [e_1, e])$ lie in $D_\theta(e)$ and it is easy to check that $\{a_0, a_1\}$ is isomorphic to U_1. So $U_1 < D_\theta(e) < D_\theta < V$. □

If θ is in class $\mathrm{III}_{N>R}$ or $\mathrm{III}_{N>N}$ we obtain a better result.

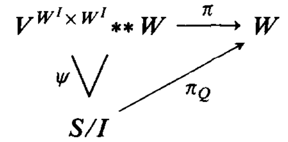
Proposition 2.11. *Let θ be in class $\mathrm{III}_{N>R}$ or $\mathrm{III}_{N>N}$. Let V be a non-trivial M-variety. There exist V in V and a division $\chi : S < V ** T$ such that $\chi\pi = \theta$. If θ is a monoid morphism, we may choose $\chi : S < V \circ\circ T$.*



Proof. Let $I = \{s \in S \mid \mathrm{not}(s >_{\mathcal{J}} J)\}$. Then, I is an ideal of S that contains J and not Q. Let us denote by π_I the canonical projection of S onto S/I. $Q \cup \{0\}$ is an ideal of S/I satisfying $(Q \cup \{0\})^2 = \{0\}$ and $(S/I)/(Q \cup \{0\}) = S/(I \cup Q)$. Let us denote $S/(I \cup Q)$ by W, by π' the projection of S onto W and by π_Q the projection of S/I onto W: $\pi' = \pi_I \pi_Q$. Since θ is one-to-one on $S \setminus (J \cup Q)$ and $Q\theta \subseteq J\theta = J'$, $s\theta = s'\theta$ implies $s\pi' = s'\pi'$ for all s, s' in S. So there exists a morphism $\varphi : T \to W$ such that $\theta\varphi = \pi'$.

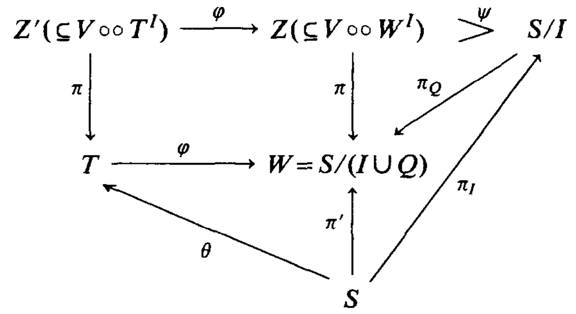


Since $(Q \cup \{0\})^2 = \{0\}$, after Proposition 1.9, there exists V in $\mathbf{V}$ and a division $\psi : S/I < V^{W^I \times W^I} ** W$, with $V^{W^I \times W^I} ** W \subsetneq V \circ\circ W^I$ such that $\psi\pi = \pi_Q$.



Also, we note that the morphism from S into $S/I \times T$ that maps s in S onto $(s\pi_I, s\theta)$ is one-to-one. Thus we can consider S as a subsemigroup of $S/I \times T$.

Let then V' be the subsemigroup of $V^{T^I \times T^I} ** T$ (where $V^{T^I \times T^I} ** T \subsetneq V \circ\circ T^I$) of all elements (f, t) such that, if u, v, u', v' are in T^I and $u\varphi = u'\varphi$, $v\varphi = v'\varphi$, then $f(u, v) = f(u', v')$. Then $\varphi : T \to W$ induces a morphism $\varphi : V' \to V^{W^I \times W^I} ** W$: $(f, t)\varphi = (g, t)$ where $g(u\varphi, v\varphi) = f(u, v)$ for all u, v in T^I. Let $Z = S/I\psi \subsetneq V^{W^I \times W^I} ** W$ and $Z' = Z\varphi^{-1} \subsetneq V' \subsetneq V^{T^I \times T^I} ** T$. We obtain the following commutative diagram:



In particular, $\pi_I\psi\pi = \pi_I\pi_Q = \pi'$ and

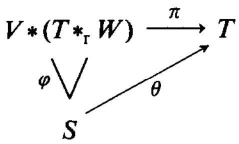
$$\pi_I\psi\varphi^{-1}\pi = \pi_I\psi\pi\varphi^{-1} = \pi_I\pi_Q\varphi^{-1} = \pi'\varphi^{-1} = \theta.$$

So the relational morphism $\chi = \pi_I\psi\varphi^{-1} : S \to Z'$ satisfies $\chi\pi = \theta$. We shall conclude by showing that χ is an injective relational morphism. Let indeed z' be in $s\chi \cap s'\chi$ $(s, s' \in S)$. Then $z'\pi \in s\chi\pi \cap s'\chi\pi = s\theta \cap s'\theta$ and hence $s\theta = s'\theta$. Furthermore, since $\varphi : Z \to Z'$ is onto, φ^{-1} is an injective relational morphism, and hence so is $\psi\varphi^{-1} : S/I < Z'$. Consequently, since $\chi = \pi_I\psi\varphi^{-1}$, $s\chi \cap s'\chi \neq \emptyset$ implies $s\pi_I = s'\pi_I$. But we noticed that $(s\pi_I, s\theta) = (s'\pi_I, s'\theta)$ implies $s = s'$. So χ is injective. □

Finally, note the following improvement on Proposition 2.8:

Proposition 2.12. *Let* $\theta : S \to T$ *be in class* $\mathrm{III}_{N>N}$, *and* $\mathbf{V}$ *and* $\mathbf{W}$ *be non-trivial*

M-varieties. There exist V *in* $\mathbf{V}$, W *in* $\mathbf{W}$ *and a division* $\varphi : S < V * (T *_{\mathrm{r}} W)$ *such that* $\varphi\pi = \theta$. *If* θ *is a monoid morphism, we may choose* $\varphi : S < V \circ (T \circ_{\mathrm{r}} W)$.

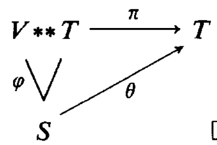


Proof. The proof is the same as for Proposition 2.8. A relational morphism $\varphi : Z \to S$ is defined as in part (a) of the proof of Proposition 2.8. Rereading part (b) of that proof, we see that we showed that the non-empty base semigroups of $D_{\varphi^{-1}}$ contain only the local identity in the case of a class $\mathrm{III}_{N>N}$ m.p.s. Indeed, if J' is null, then we cannot have simultaneously $(s_n \cdots s_1)\theta \in J'$ and $s'\theta = (s'_p \cdots s'_1)\theta \in J'$. □

2.4. *Class* IV

Let $\theta : S \to T$ be in class IV. After Proposition 2.4 and the remark that followed, we have

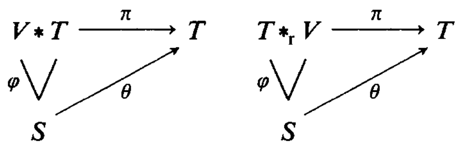
Proposition 2.13. *If* $\mathbf{V}$ *is a non-trivial M-variety, there exist* V *in* $\mathbf{V}$ *and a division* $\varphi : S < V ** T$ *such that* $\varphi\pi = \theta$. *If* θ *is a monoid morphism,* φ *can be chosen* $S < V \circ\circ T$.



□

In fact, we have a better result.

Proposition 2.14. *If* $\mathbf{V}$ *is a non-trivial M-variety, there exist* V *in* $\mathbf{V}$ *and a division* $\varphi : S < V * T$ *(resp.* $S < T *_{\mathrm{r}} V$*) such that* $\varphi\pi = \theta$. *If* θ *is a monoid morphism,* φ *may be chosen* $S < V \circ T$ *(resp.* $S < T \circ_{\mathrm{r}} V$*).*



Proof. As in several above proofs, it is enough to show that the non-empty base semigroups of D_θ contain only one element, which is the local identity. Let t_1 be an object of D_θ such that $D_\theta(t_1) \neq \emptyset$. If $t_1 \notin J'$, $t_1\theta^{-1}$ has one element, the only mapping from $t_1\theta^{-1}$ into itself is the identity, and hence $D_\theta(t_1)$ contains only the local identity. Otherwise, $t_1 \in J'$ and $t_1\theta^{-1} = \{q_1, s_1\}$ with $q_1 \in Q$ and $s_1 \in J$. Let then $a = (t_1, [s, t]) \in$

$D_\theta(t_1)$. q_1s and s_1s are still in $t_1\theta^{-1}$. Also, $q_1s \leq_{\mathscr{J}} q_1$, $s_1s \leq_{\mathscr{J}} s_1$ and, since J and Q are not $\mathscr{J}$-comparable, $q_1s = q_1$ and $s_1s = s_1$. So a induces the identity on $t_1\theta^{-1}$ and hence is the local identity.

The statement relative to $*_r$ and $\circ_r$ is obtained by considering θ^r, which is also in class IV. □

2.5. Summary

Let $\theta: S \to T$ be an m.p.s. We shall summarize the results of Subsections 2.1 to 2.4 in Table 1. $\boldsymbol{V}$ and $\boldsymbol{W}$ denote non-trivial $\boldsymbol{M}$-varieties. The notation $S < \boldsymbol{V} * T$ (resp. $S < T *_r \boldsymbol{V}$, $S < \boldsymbol{V} ** T$) means that there exist V in $\boldsymbol{V}$ and a division $\varphi: S < V * T$ (resp. $S < T *_r V$, $S < V ** T$) such that $\varphi\pi = \theta$.

Let us note that, for m.p.s.'s in class $\mathrm{III}_{N>R}$, we have not obtained any decomposition result using $*$ or $*_r$ that would be tighter than the one that holds in general for class III m.p.s.'s. In particular, we have no result allowing us to decompose a class $\mathrm{III}_{N>R}$ m.p.s. with semidirect product and reverse semidirect product by elements of some *arbitrary* $\boldsymbol{M}$-variety, while it is possible if one uses the 2-sided semidirect product. We shall elaborate on these remarks in Subsection 3.3.

Table 1

Class of m.p.s.'s	Decomposition with **	Decomposition with * and $*_r$
I and I_R	$S < \boldsymbol{G} ** T$	$S < \boldsymbol{G} * T$ and $S < T *_r \boldsymbol{G}$
I_N	$S < \boldsymbol{V} ** T$	$S < \boldsymbol{V} * (T *_r \boldsymbol{W})$ + dual
$\mathrm{II}_{\mathrm{row}}$	$S < \boldsymbol{V} ** T$	$S < \boldsymbol{V} * T$
$\mathrm{II}_{\mathrm{col}}$	$S < \boldsymbol{V} ** T$	$S < T *_r \boldsymbol{V}$
III and $\mathrm{III}_{R>R}$	$S < \boldsymbol{J}_1 ** (\boldsymbol{W} ** T)$	$S < \boldsymbol{J}_1 * (T *_r \boldsymbol{W})$
		$S < (\boldsymbol{W} * T) *_r \boldsymbol{J}_1$
$\mathrm{III}_{N>R}$	$S < \boldsymbol{V} ** T$	
$\mathrm{III}_{N>N}$	$S < \boldsymbol{V} ** T$	$S < \boldsymbol{V} * (T *_r \boldsymbol{W})$ + dual
IV	$S < \boldsymbol{V} ** T$	$S < \boldsymbol{V} * T$ and $S < T *_r \boldsymbol{V}$

3. Decomposition of morphisms and varieties

We shall use the above results to prove new results or give new proofs of known results concerning the decomposition of semigroups and morphisms.

Note the following notational convention: if $V_n, \ldots, V_1$ are semigroups (resp. varieties) and T is a semigroup, $V_n ** (V_{n-1} ** (\cdots (V_1 ** T) \cdots))$ will be denoted by $V_n ** V_{n-1} ** \cdots ** V_1 ** T$ and $V_n * (V_{n-1} * (\cdots (V_1 * T) \cdots))$ by $V_n * V_{n-1} * \cdots * V_1 * T$. That is, the order in which the products are performed is implicitly considered to be from right to left.

3.1. The prime decomposition theorem

We give a modern proof of the Krohn–Rhodes theorem.

Let S and T be semigroups and let $\mathbf{H}$ be the $\mathbf{G}$-variety generated by the groups in S. If $\beta: S \to T$ is an m.p.s., we showed in Subsection 2.5 that there exist M in $\mathbf{J}_1$ or $\mathbf{H}$ and a division $\varphi: S < M ** T$ such that $\varphi\pi = \theta$. Consequently, if $\beta: S \to T$ is any onto morphism, there exist $M_n, \ldots, M_1$ in $\mathbf{J}_1$ or $\mathbf{H}$ and a division $\varphi: S < M_n ** \cdots ** M_1 ** T$ such that $\varphi\pi = \theta$: this is obtained by considering a factorization of β in m.p.s.'s. Finally, if $\tau: S \to T$ is an onto relational morphism, let $R = \{(s,t) \in S \times T \mid t \in s\tau\}$, and α and β be the projections of R onto S and T. R is a subsemigroup of $S \times T$ and α and β are onto morphisms such that $\tau = \alpha^{-1}\beta$. Since α^{-1} is a division $S < R$, there exist $M_n, \ldots, M_1$ in $\mathbf{J}_1$ or $\mathbf{H}$ and a division $\varphi: S < M_n ** \cdots ** M_1 ** T$ such that $\varphi\pi = \theta$. So we proved

Proposition 3.1. *If $\tau: S \to T$ is an onto relational morphism, $S < \mathbf{V}_n ** \cdots ** \mathbf{V}_1 ** T$ where $\mathbf{V}_i$ $(1 \le i \le n)$ is an $\mathbf{M}$-variety equal to $\mathbf{J}_1$ or $\mathbf{H}$. Furthermore, if S, T are monoids and $1 \in 1\tau$, then $S < M_n \circ\circ \cdots \circ\circ M_1 \circ\circ T$ with the M_i's $(1 \le i \le n)$ in $\mathbf{J}_1$ or $\mathbf{H}$.* □

Note that, if M is a monoid, $M ** 1 = M$. By applying Proposition 3.1 to morphisms of the form $\tau: S \to 1$, we obtain

Corollary 3.2. *Let S be any semigroup. Then $S < \mathbf{V}_n ** \cdots ** \mathbf{V}_1 ** 1$, i.e. $S \in (\mathbf{V}_n)_S ** \cdots ** (\mathbf{V}_1)_S ** \mathbf{T}$ where $\mathbf{V}_i$ $(1 \le i \le n)$ is either $\mathbf{J}_1$ or $\mathbf{H}$, $\mathbf{T} = \{\emptyset, \{1\}\}$ is the trivial $\mathbf{S}$-variety and $(\mathbf{V}_i)_S$ is the $\mathbf{S}$-variety generated by $\mathbf{V}_i$. If S is a monoid, $S \in \mathbf{V}_n ** \cdots ** \mathbf{V}_1$.* □

This can be rephrased as follows:

Corollary 3.3. *The least $\mathbf{S}$-variety (resp. $\mathbf{M}$-variety) containing $\mathbf{J}_1$ and $\mathbf{G}$ and closed under $**$ is the variety of all semigroups (resp. monoids). The same holds if we replace the closure under $**$ by the closure under both $*$ and $*_r$.*

Proof. The first statement is a rewriting of Corollary 3.2. The second one is a consequence of the fact that $S ** T \subseteq T *_r (S * T)$ (see Subsection 1.2). □

3.2. Aperiodic and **LG**-*morphisms*

After [10, Proposition 3.8], we know that an onto morphism $\beta: S \to T$ is aperiodic (resp. an **LG**-morphism) iff no m.p.s. occuring in a factorization of β is in class I_R (resp. $\mathrm{III}_{R>R}$).

Proposition 3.4. *Let $\tau: S \to T$ be an onto relational morphism.*

(1) *τ is aperiodic iff $S < \mathbf{J}_1 ** \cdots ** \mathbf{J}_1 ** T$. If S and T are monoids and $1 \in 1\tau$, $S < M_n ** \cdots ** M_1 ** T$ where $M_i \in \mathbf{J}_1$ $(1 \le i \le n)$.*

(2) *Let* $\mathbf{H}$ *be a non-trivial* G*-variety. If* τ *is an* $\mathbf{LH}$*-relational morphism,* $S < \mathbf{H} ** \cdots ** \mathbf{H} ** T$. *If* S *and* T *are monoids and* $1 \in 1\tau$, $S < H_n ** \cdots ** H_1 ** T$ *where* $H_i \in \mathbf{H}$ $(1 \le i \le n)$. *Conversely, if* $S < \mathbf{G} ** \cdots ** \mathbf{G} ** T$, *then* τ *is an* $\mathbf{LG}$*-morphism.*

Proof. Let $\tau = \alpha^{-1}\beta$ be the factorization of τ as in Subsection 3.1. Since α^{-1} is a division, it is enough to prove the proposition in the case of an onto (functional) morphism $\beta : S \to T$. Let $\beta = \theta_1 \cdots \theta_n$ be a factorization of β in m.p.s.'s with $\theta_i : S_i \to S_{i+1}$ $(1 \le i \le n)$, $S_1 = S$ and $S_{n+1} = T$. If β is aperiodic, none of the θ_i's is in class I_R and hence, after Proposition 2.5, $S_i < \mathbf{J}_1 ** S_{i+1}$ for all $1 \le i \le n$. Thus, $S < \mathbf{J}_1 ** \cdots ** \mathbf{J}_1 ** T$.

If β is an $\mathbf{LH}$-morphism, it is an $\mathbf{LG}$-morphism after [10, Proposition 1.1(2)], so that none of the θ_i's $(1 \le i \le n)$ is in class $\mathrm{III}_{R>R}$. Let us now examine each θ_i in turn. If θ_i is aperiodic, $S_i < \mathbf{V} ** S_{i+1}$ for every non-trivial M-variety $\mathbf{V}$, and in particular, $S_i < \mathbf{H} ** S_{i+1}$. If θ_i is not aperiodic, $S_i < \mathbf{H}_i ** S_{i+1}$ where $\mathbf{H}_i$ is the G-variety generated by the groups dividing some $e_{i+1}\theta_i^{-1}$, for all idempotents e_{i+1} in S_{i+1}. But $e_{i+1}\theta_i^{-1} \subseteq e_{i+1}(\theta_{i+1} \cdots \theta_n)\beta^{-1}\theta_1 \cdots \theta_{i-1} < e_{i+1}(\theta_{i+1} \cdots \theta_n)\beta^{-1}$. Since β is an $\mathbf{LH}$-morphism, $e_{i+1}(\theta_{i+1} \cdots \theta_n)\beta^{-1}$ is in $\mathbf{LH}$ and hence so is $e_{i+1}\theta_i^{-1}$. Consequently, the groups in $e_{i+1}\theta_i^{-1}$ are in $\mathbf{H}$ and $S_i < \mathbf{H} ** S_{i+1}$. Thus $S < \mathbf{H} ** \cdots ** \mathbf{H} ** T$.

Conversely, let $\varphi : S < M_n ** \cdots ** M_1 ** T$ be such that $\beta = \varphi\pi$ and the M_i's are in $\mathbf{J}_1$ (resp. in $\mathbf{G}$). Let also π_i denote the projection from $M_i ** \cdots ** M_1 ** T$ onto $M_{i-1} ** \cdots ** M_1 ** T$. Then $\beta = \varphi\pi_n \cdots \pi_1$. But φ is injective and hence trivially an aperiodic and an $\mathbf{LG}$-morphism. So, it is enough to show that, if $S \in \mathbf{A}$ (resp. $S \in \mathbf{LG}$), then the projection π from $S ** T$ onto T is aperiodic (resp. an $\mathbf{LG}$-morphism). Then β will be a composition of aperiodic (resp. $\mathbf{LG}$-morphism) morphisms, and hence will be aperiodic (resp. an $\mathbf{LG}$-morphism) itself. Let T' be an aperiodic subsemigroup (resp. a subsemigroup in $\mathbf{LG}$) of T. Then $T'\pi^{-1} = S ** T' \in \mathbf{A} ** \mathbf{A} \subseteq \mathbf{A} *_{\mathrm{r}} (\mathbf{A} * \mathbf{A})$ (resp. $\in \mathbf{LG} ** \mathbf{LG} \subseteq \mathbf{LG} *_{\mathrm{r}} (\mathbf{LG} * \mathbf{LG})$). But it is known that $\mathbf{A}$ and $\mathbf{LG}$ are closed under $*$ and $*_{\mathrm{r}}$. So $T'\pi^{-1} \in \mathbf{A}$ (resp. $\mathbf{LG}$). □

Note that in [6], it was proved for arbitrary semigroups that, if $\tau : S \to T$ is an aperiodic relational morphism and $\mathbf{V}$ is a non-trivial M-variety, then $S < \mathbf{A} * (T *_{\mathrm{r}} \mathbf{V})$. If we apply Proposition 3.4 to the morphism $\beta : S \to 1$, we obtain the following corollary:

Corollary 3.5. (1) *The least* S*-variety and* M*-variety containing* $\mathbf{J}_1$ *and closed under* $**$ *(resp. closed under both* $*$ *and* $*_{\mathrm{r}}$*) are* $\mathbf{A}_S$ *and* $\mathbf{A}$.

(2) *The least* S*-variety containing* $\mathbf{G}$ *and closed under* $**$ *is* $\mathbf{LG}$. □

Recall that a relational morphism $\tau : S \to T$ is $\mathbf{LI}$ iff it is both aperiodic and $\mathbf{LG}$ (as a consequence of [10, Proposition 1.1]).

Proposition 3.6. *Let* $\tau : S \to T$ *be an onto relational morphism and* $\mathbf{V}$ *be a non-trivial* M*-variety. If* τ *is an* $\mathbf{LI}$*-relational morphism, then* $S < \mathbf{V} ** \cdots ** \mathbf{V} ** T$. *Moreover,* *if S and T are monoids, and $1 \in 1\tau$, then $S < M_n \circ\circ \cdots \circ\circ M_1 \circ\circ T$ with $M_i \in V$ $(1 \le i \le n)$.*

Proof. As in the proof of Proposition 3.4, it is enough to consider the case of a (functional) morphism $\beta : S \to T$. Let $\beta = \theta_1 \cdots \theta_n$ be a factorization of β in m.p.s.'s. After [9, Proposition 3.8], none of the θ_i's is in class I_R or $III_{R>R}$. Thus, if θ_i maps S_i onto S_{i+1} $(1 \le i \le n,\ S_1 = S,\ S_{n+1} = T)$, then $S_i < V ** S_{i+1}$ after Subsection 2.5. So $S < V ** \cdots ** V ** T$. □

For each non-trivial M-variety V, let V_c denote the least S-variety containing V and closed under $**$.

Corollary 3.7. $\mathbf{LI} = \bigcap V_c$, *where the intersection ranges over all non-trivial M-varieties V.*

Proof. If we apply Proposition 3.6 to the morphism $\beta : S \to 1$ for $S \in \mathbf{LI}$, we obtain the inclusion $\mathbf{LI} \subseteq V_c$ for all V. The converse is a consequence of the facts that $A_c = A_S$ and that $G_c = \mathbf{LG}$. These facts were proved in Corollary 3.5. □

3.3. *Regular* LG-*morphisms*

Recall that $\theta : S \twoheadrightarrow T$ is a regular surmorphism iff $\theta(s)$ is regular iff s is regular. Regular morphisms, i.e. morphisms that are both **LG**-morphisms and regular, were studied in particular in [8] under the name of $\mathcal{J}^*$-morphisms.

Let us first set the following definition: if V is a class of monoids, and S and T are semigroups, we say that S is a *multiple product* of T by elements of V if there exists a sequence $(S_i)_{0 \le i \le n}$ of semigroups such that $S_0 = T$, $S_n = S$, and for all $1 \le i \le n$, S_i is either of the form $V_i * S_{i-1}$ or of the form $S_{i-1} *_r V_i$, with the V_i's in V. Note that there still exists a canonical projection $\pi : S \to T$.

After [10, Proposition 3.8] and the Appendix of [8], we know that a morphism $\beta : S \to T$ is a regular **LG**-morphism (resp. a regular **LI**-morphism) iff no m.p.s. occuring in a factorization of β is in class $III_{R>R}$ nor in class $III_{N>R}$ (resp. in class $III_{R>R}$, nor in class $III_{N>R}$, nor in class I_R). In view of Subsection 2.5, Proposition 2.1 and the proof of Proposition 3.4, this proves the following:

Proposition 3.8. *Let $\beta : S \to T$ be a regular* **LG**-*morphism and V be any non-trivial M-variety containing the subgroups of S. Then there exists a multiple product V of T by elements of V and a division $\varphi : S < V$ such that $\varphi\pi = \beta$. If β is a regular* **LI**-*morphism, V can be chosen to be any non-trivial M-variety.* □

Note that the above proposition holds in particular for $V = G$. In that case, the converse holds too.

Proposition 3.9. *Let S and T be semigroups, V be a multiple product of T by groups* *and $\varphi : S < V$ be a division, such that $\beta = \varphi\pi : S \to T$ is a (functional) morphism. Then β is a regular* **LG**-*morphism.*

Proof. By definition, there exist semigroups $V_0, \ldots, V_n$ such that $V_0 = T$, $V_n = V$ and, for each $1 \le i \le n$, $V_i = G_i * V_{i-1}$ or $V_{i-1} *_{\mathrm{r}} G_i$ for some group G_i. Also there exists a subsemigroup W of V and an onto morphism $\psi : W \to S$ such that $\varphi = \psi^{-1}$, and hence $\psi\beta$ is the restriction of π to W. In particular, $W\pi = T$.

Note that a morphism is a regular **LG**-morphism iff the inverse image of every regular $\mathcal{J}$-class is a regular $\mathcal{J}$-class. So, it is enough to show that $\pi : W \to T$ is a regular **LG**-morphism. Indeed, in that case, if J' is a regular $\mathcal{J}$-class of T, then $J'\beta^{-1} = (J'\pi^{-1} \cap W)\psi$ is the image of a regular $\mathcal{J}$-class of W, and hence a regular $\mathcal{J}$-class.

Let us first prove that $\pi : V \to T$ is a regular **LG**-morphism. Since π is the product of the $\pi_i : V_i \to V_{i-1}$ $(1 \le i \le n)$, we need to show that a projection of the form $\pi : S *_{\mathrm{r}} G \to S$ or $\pi : G * S \to S$ (with G a group) is a regular **LG**-morphism. Let us consider $\pi : G * S \to S$ (the other case is dual). We shall use for G an additive notation, without assuming commutativity. Let s be a regular element of S, and $(g, s) \in s\pi^{-1}$. Then there exists an idempotent e and an element s' of S such that $s's = e$ and $se = s$. Let $g' = -s' \cdot g$. Then $(g', s')(g, s) = (g' + s' \cdot g, s's) = (0, e)$, $(g, s)(0, e) = (g + s \cdot 0, se) = (g + 0, s) = (g, s)$ and $(0, e)^2 = (0, e)$. So (g, s) is $\mathcal{L}$-equivalent to some idempotent of $G * S$ and hence is regular. Thus π is a regular morphism. In order to prove that π is also a **LG**-morphism, we need to show that, for all idempotents e of S, $e\pi^{-1} \in$ **LG**. Let (g, e) be an idempotent of $e\pi^{-1}$. In particular $g + e \cdot g = g$, so that $e \cdot g = 0$. For all $h \in G$, $(g, e)(h, e)(g, e)(-h, e)(g, e) = (g + e \cdot h + e \cdot g - e \cdot h + e \cdot g, e) = (g, e)$. So $(g, e)e\pi^{-1}(g, e)$ is a group, which makes $e\pi^{-1}$ an element of **LG**.

Finally, we need to show that the restriction $\pi : W \to T$ is a regular **LG**-morphism. Let J' be a regular $\mathcal{J}$-class of T. After the above discussion, $J'\pi^{-1} = J$ is a regular $\mathcal{J}$-class of V. Also, since $W\pi = T$, $(J \cap W)\pi = J'$. Let $w \in J \cap W$. There exists an idempotent e in J and an element w_1 in J such that $e = ww_1$. Since $w_1\pi \in J'$, we can pick w_2 in $J \cap W$ such that $w_2\pi = w_1\pi$. Then $(ww_2)\pi = e\pi$ is idempotent and, for some n, $e' = (ww_2)^n$ is an idempotent of W such that $e'\pi = e\pi$. Consequently $e' \le_{\mathcal{R}} w$ in W and $e' \in J \cap W$. Thus, each element w of $J \cap W$ is $\mathcal{R}$-equivalent, in W, to some idempotent and hence is a regular element of W. So, $J \cap W = J'\pi^{-1} \cap W$ is a regular $\mathcal{J}$-class of W, which concludes the proof. □

On the other hand, if N is a non-trivial ideal of a semigroup S such that $N^2 = \{0\}$, then $\pi : S \to S/N$ is not regular. In particular, after Proposition 3.9, π cannot be decomposed through a multiple product of S/N by groups. However, we know that this same morphism can be decomposed through a single 2-sided product $G ** S/N$ for any suitably large group G (see Proposition 1.9).

This remark makes apparent the main difference between the decomposing powers of the 2-sided product on one hand, and the multiple product, i.e. a combination of semidirect and reverse semidirect products, on the other hand.

It is also of interest to note the following result, whose proof relies on Brown's lemma [2].

Lemma 3.10 (Brown). *Let S and T be semigroups, possibly infinite, and $\beta : S \to T$ be a morphism. If T is locally finite and, for each idempotent e of T, $e\beta^{-1}$ is locally finite, then S is locally finite.* □

Proposition 3.11. *Let T be a finite semigroup and $n \geq 1$. The cardinality of the n-generated semigroups S such that there exists a regular* **LI**-*morphism $\beta : S \to T$ is bounded.*

Proof. Let A be a n-letter alphabet and S be a possibly infinite A-generated semigroup. Let $\beta : S \to T$ be aperiodic and such that the inverse image of a regular $\mathcal{J}$-class is a regular $\mathcal{J}$-class. For each idempotent e of T, $e\beta^{-1}$ is an aperiodic regular simple semigroup, i.e. a rectangular band. But rectangular bands form a locally finite variety, so that $e\pi^{-1}$ is locally finite. By Brown's lemma, this implies that S is finite.

For each morphism $\pi : A^+ \to T$ (and there are only finitely many such morphisms), let $(S_i, \beta_i, \sigma_i)_{i \in I}$ be the family of all triples (S, β, σ) where S is a finite semigroup, $\sigma : A^+ \to S$ is an onto morphism, $\beta : S \to T$ is a regular **LI**-morphism, and $\tau = \sigma\beta$. Note that each semigroup S for which there exists a regular **LI**-morphism from S into T is an S_i for some choice of τ.

Let then S be the subsemigroup of $\prod_{i \in I} S_i$ generated by the $(a\sigma_i)_{i \in I}$, for all $a \in A$, and let $\beta : S \to T$ be the restriction of $(\beta_i)_{i \in I}$ to S. Since each β_i is a regular **LI**-morphism, the reverse image of any regular $\mathcal{J}$-class of T by β is a regular $\mathcal{J}$-class of S (see the proof of Proposition 3.9 above), and hence S is finite. This is to say that there exists a maximal (finite) object $(S, \beta, (\sigma_i)_{i \in I})$ in the family $(S_i, \beta_i, \sigma_i)_{i \in I}$, which concludes the proof. □

Note that the hypothesis that the morphisms are regular is essential. Indeed, if S is any cyclic aperiodic semigroup, then $\beta : S \to 1$ is an **LI**-morphism, but S can have an arbitrarily cardinality. Recall that, after Proposition 3.6, **LI**-morphisms can be decomposed by 2-sided products by V. So, given T and a set of generators for S, if $\beta : S \to T$ is a morphism that can be decomposed through $V ** \cdots ** V ** T$, then S can be arbitrarily large, while if β is decomposed through a multiple product of T by V, the cardinality of S is bounded. As before, V denotes any non-trivial variety.

3.4. Other applications

Let $\theta : S \to T$ be an m.p.s., Subsection 2.5 shows that $S < V * T$, for any non-trivial M-variety V, iff θ is neither in class I, nor III, nor $\mathrm{II}_{\mathrm{col}}$. We know that θ is not in class I iff it is injective on $\mathcal{H}$-classes. The exclusion of class III is equivalent to the following condition:

(P) If $x <_{\mathcal{J}} y$, then $x\theta <_{\mathcal{J}} y\theta$.

Finally, if θ is injective on $\mathcal{H}$-classes, then it is not in class $\mathrm{II}_{\mathrm{col}}$ iff it is injective on $\mathcal{R}$-classes.

Note that, if $\beta = \beta_1\beta_2$, where $\beta_1 : S \to V$ and $\beta_2 : V \to T$ are onto morphisms, then β is injective on $\mathcal{R}$-classes iff both β_1 and β_2 are.

The same holds for property (P) as well, as we now prove. If β_1 and β_2 satisfy (P), it is trivial that β does too. Let us now prove the converse. Let $s, s' \in S$ satisfy $s <_{\mathcal{J}} s'$. Then $s\beta_1 \leq_{\mathcal{J}} s'\beta_1$. But $s\beta_1 \mathcal{J} s'\beta_1$ would imply that $s\beta = (s\beta_1)\beta_2$ and $s'\beta = (s'\beta_1)\beta_2$ are $\mathcal{J}$-equivalent, which is absurd. So β_1 satisfies (P). Let now $v, v' \in V$ such that $v <_{\mathcal{J}} v'$, and $s' \in v'\beta_1^{-1}$. Then $v = w_1 v' w_2$ for some $w_1, w_2 \in V^{\bullet}$. Let then $s_1 \in w_1\beta_1^{-1}$ and $s_2 \in w_2\beta_1^{-1}$. We have $s = s_1 s' s_2 \in v\beta_1^{-1}$ and $s \leq_{\mathcal{J}} s'$. But s cannot be $\mathcal{J}$-equivalent to s' since their β_1-images v and v' are not $\mathcal{J}$-equivalent. So $s <_{\mathcal{J}} s'$ and hence $v\beta_2 = s\beta <_{\mathcal{J}} s'\beta = v'\beta_2$. β_2 also satisfies (P).

Let us then consider an onto morphism $\beta : S \to T$ that satisfies (P) and is injective on $\mathcal{R}$-classes, and let $\beta = \theta_1 \cdots \theta_n$ be a factorization of β in m.p.s.'s, with $\theta_i : S_i \to S_{i+1}$, $S_1 = S$ and $S_{n+1} = T$. Then, each θ_i $(1 \leq i \leq n)$ satisfies (P) and is injective on $\mathcal{R}$-classes and hence $S_i < \boldsymbol{V} * S_{i+1}$. This proves the following:

Proposition 3.12. *Let $\beta : S \to T$ be an onto morphism satisfying* (P) *and injective on $\mathcal{R}$-classes and let $\boldsymbol{V}$ be a non-trivial M-variety. Then $S < \boldsymbol{V} * \cdots * \boldsymbol{V} * T$. Furthermore, if S and T are monoids, $S < M_n \circ \cdots \circ M_1 \circ T$ with $M_i \in \boldsymbol{V}$ $(1 \leq i \leq n)$.*

Note the following particular case:

Corollary 3.13. *Let $\beta : S \to T$ be an onto morphism satisfying* (P) *and injective on $\mathcal{R}$-classes. If $\boldsymbol{V}$ is an S- or an M-variety and $T \in \boldsymbol{V}$, then $S \in \boldsymbol{R} * \boldsymbol{V}$.* □

Proof. By the above theorem, $S \in \boldsymbol{J}_1 * \cdots * \boldsymbol{J}_1 * \boldsymbol{V}$. But $*$ is associative, $\boldsymbol{J}_1 \subseteq \boldsymbol{R}$ and $\boldsymbol{R} * \boldsymbol{R} = \boldsymbol{R}$ [11], so that $S \in \boldsymbol{R} * \boldsymbol{V}$. □

Similarly, if $\theta : S \to T$ is an m.p.s. that is neither in class III nor in class $\mathrm{II}_{\mathrm{col}}$, then $S < \boldsymbol{G} * T$ and, by the results of Subsection 2.1, $S < \boldsymbol{H} * T$ for any non-trivial $\boldsymbol{G}$-variety $\boldsymbol{H}$ containing the groups of S.

As above, one can check that the exclusion of classes III and $\mathrm{II}_{\mathrm{col}}$ is equivalent to condition (P) and condition (Q) below.

(Q) If $x \mathcal{R} y$ and $x\theta = y\theta$, then $x\theta \mathcal{H} y\theta$.

Again we need to check that, if onto morphisms $\beta_1 : S \to V$ and $\beta_2 : V \to T$ are such that $\beta = \beta_1\beta_2$ satisfies (Q), then so do β_1 and β_2. Let $s, s' \in S$ be such that $s \mathcal{R} s'$ and $s\beta_1 = s'\beta_1$. Then $s\beta = s'\beta$ and, since β satisfies (Q), we have $s \mathcal{H} s'$. Let now $v, v' \in V$ satisfy $v \mathcal{R} v'$ and $v\beta_2 = v'\beta_2$. Then $v = v'a'$ and $v' = vb'$ for some a', b' in V. Let $s \in v\beta_1^{-1}$, $a \in a'\beta_1^{-1}$ and $b \in b'\beta_1^{-1}$. Then for all $i \geq 0$, $s(ba)^i\beta_1 = v$ and $s(ba)^i b\beta_1 = v'$.

If i is such that $(ba)^i$ is idempotent, we see that $s(ba)^i$ and $s(ba)^i b$ are $\mathscr{R}$-equivalent. But their β-images are equal so that they are $\mathscr{H}$-equivalent. Thus $v\mathscr{H}v'$.

So we have proved the following:

Proposition 3.14. *Let $\beta : S \to T$ be an onto morphism satisfying* (P) *and* (Q) *and let $\boldsymbol{H}$ be a non-trivial $\boldsymbol{G}$-variety containing the groups of S. Then $S < \boldsymbol{H} * \cdots * \boldsymbol{H} * T$. In particular, if $\boldsymbol{H} * \boldsymbol{H} = \boldsymbol{H}$ and $T \in \boldsymbol{V}$, then $S \in \boldsymbol{H} * \boldsymbol{V}$. If S and T are monoids, $S < M_n \circ \cdots \circ M_1 \circ T$ with $M_i \in \boldsymbol{H}$ $(1 \le i \le n)$.* □

Of course, the dual statements of Propositions 3.12 and 3.14 obtained by replacing $\mathscr{R}$ by $\mathscr{L}$ hold as well.

Proposition 3.15. (1) *The least $\boldsymbol{S}$-variety* (*resp. $\boldsymbol{M}$-variety*) *containing $\boldsymbol{J}_1$ and closed under $*_\mathrm{r}$ is $\boldsymbol{R}_S^\mathrm{r}$* (*resp. $\boldsymbol{R}^\mathrm{r}$*).

(2) *If $\beta : S \to T$ is an onto morphism injective on $\mathscr{L}$-classes and $\boldsymbol{V}$ is an $\boldsymbol{S}$- or an $\boldsymbol{M}$-variety containing T, then $S \in \boldsymbol{V} *_\mathrm{r} \boldsymbol{R}^\mathrm{r}$.* □

Finally, Subsection 2.5 shows that if $\theta : S \to T$ is an m.p.s., $S < \boldsymbol{G} * T$ iff θ is neither in class III nor class $\mathrm{II}_{\mathrm{col}}$. This is the case in particular if θ is injective on $\mathscr{R}$-classes and satisfies (P). Indeed, (P) prevents θ from being in class III. Note that the injectivity on $\mathscr{R}$-classes makes θ aperiodic so that $S < \boldsymbol{H} * T$ for any non-trivial $\boldsymbol{G}$-variety $\boldsymbol{H}$.

Property (P) is such that, if $\beta_1 : S \to V$ and $\beta_2 : V \to T$ are onto morphisms, and $\beta = \beta_1\beta_2$ satisfies (P), then both β_1 and β_2 satisfy (P). So, if $\beta : S \to T$ is an onto morphism that is injective on $\mathscr{R}$-classes and satisfies (P), and $\beta = \theta_1 \cdots \theta_n$ is a factorization in m.p.s.'s, then each θ_i is injective on $\mathscr{R}$-classes and satisfies (P). This proves the following proposition:

Proposition 3.16. *Let $\beta : S \to T$ be an onto morphism injective on $\mathscr{R}$-classes* (*resp. on $\mathscr{L}$-classes*) *and satisfying* (P), *and let $\boldsymbol{H}$ be a non-trivial $\boldsymbol{G}$-variety. Then $S < \boldsymbol{H} * \cdots * \boldsymbol{H} * T$* (*resp. $S < T *_\mathrm{r} \boldsymbol{H} *_\mathrm{r} \cdots *_\mathrm{r} \boldsymbol{H}$*). *In particular, if $\boldsymbol{H} * \boldsymbol{H} = \boldsymbol{H}$ and $T \in \boldsymbol{V}$, then $S \in \boldsymbol{H} * \boldsymbol{V}$* (*resp. $S \in \boldsymbol{V} *_\mathrm{r} \boldsymbol{H}$*). *If β is a monoid morphism, $S < H_n \circ \cdots \circ H_1 \circ T$* (*resp. $S < T \circ_\mathrm{r} H_1 \circ_\mathrm{r} \cdots \circ_\mathrm{r} H_n$*) *with $H_i \in \boldsymbol{H}$ $(1 \le i \le n)$.* □

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